# Midterm #1

**Full Name**: ....................................................................... **Signature**............................................

**Note**: You need to give **Justification** to all your **Answers** in order to have full **CREDIT**.

## Exercise 1. *(25 points)*

1. *Let $\alpha(n) = n^3 - 1$ be defined from $\mathbb{Z}$ to $\mathbb{Z}$*
   (a) *is $\alpha$ 1 to 1 ?*

   (b) *is $\alpha$ onto ?*

   (c) *is $\alpha$ invertible?*

2. *Let $\alpha$ and $\beta$ be defined from $S$ to $T$. Complete the following statements*
   (a) *$\alpha$ is not one to one iff* ..................................................................

   (b) *$\alpha$ is not onto iff* ..........................................................................

   (c) *$\beta \neq \alpha$ iff* .....................................................................................

**Exercise 2.** *(25 points)*

1. *Let* $\alpha : \mathbb{Z} \to \mathbb{Z}$ *defined by* $\alpha(n) = n+1$ *and* $\beta : \mathbb{Z} \to \mathbb{Z}$ *defined by* $\beta(n) = [\alpha(n)]^2$

   (a) *Write the formula for* $\alpha \circ \alpha$

   (b) *is* $\alpha \circ \alpha = \beta$ *?*

2. *Prove that if* $\beta : S \to T$, $\gamma : S \to T$, $\alpha : T \to U$, $\alpha$ *is one to one and* $\alpha \circ \beta = \alpha \circ \gamma$ *then* $\beta = \gamma$

**Exercise 3.** *(25 points)*

1. *Let* $(H, \star)$ *and* $(K, \dagger)$ *be two groups, define*
   $G = H \times K = \{(h,k); h \in H \quad \text{and} \quad k \in K\}$,
   *and define on* $G$ *the operation* $\otimes$ *by* $(h_1, k_1) \otimes (h_2, k_2) = (h_1 \star h_2, k_1 \dagger k_2)$.
   *Show that* $(G, \otimes)$ *is a group. (You can assume that* $\otimes$ *is associative in* $G$*).*

2. *Let* $G = \{z \in \mathbb{C}; z^n = 1, \text{ for some positive integrer } n\}$, *show that* $(G, \cdot)$ *is a group under multiplication.*

**Exercise 4.** *(25 points)*

1. *Let* $\alpha = (13)(24)$ *and* $\beta = (1423)$ *be two permutations in* $S_4$. *Compute the following permutations*

   *(a)* $\alpha \circ \beta$

   *(b)* $(\alpha \circ \beta)^{-1}$

   *(c)* $\alpha^{-1} \circ \beta^{-1}$

2. *Determine the group of symmetries of a rectangle.*

3. *Prove that if* $a \equiv b \pmod{n}$ *and* $c \equiv d \pmod{n}$, *then* $a + c \equiv b + d \pmod{n}$.

**Exercise 5.** *(Bonus: 20 points)*
*Prove that if* $p$ *and* $q$ *are two distinct primes numbers and* $n = pq$*, then* $(\mathbb{Z}_n^\star, \odot)$ *is not a group. (* $\mathbb{Z}_n^\star = \mathbb{Z}_n - \{[0]\}$ *and* $[x] \odot [y] = [x \cdot y]$*)*