Edgeworth Expansion of the Largest Eigenvalue Distribution Function of GOE

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Abstract

In this paper we focus on the large n probability distribution function of the largest eigenvalue in the Gaussian Orthogonal Ensemble of $n \times n$ matrices (GOE_n). We prove an Edgeworth type Theorem for the largest eigenvalue probability distribution function of GOE_n. The correction terms to the limiting probability distribution are expressed in terms of the same Painlevé II functions appearing in the Tracy-Widom distribution. We conclude with a brief discussion of the GSE_n case.

1 Introduction

Limiting probability distributions laws from Random Matrix Theory have found many applications outside their initial domain of discovery; the length of the longest increasing subsequence (P. Deift et al. [1]) properly scaled converges in distribution to the Unitary Tracy-Widom law, the properly scaled largest principal component of a white Wishart converges in distribution to the Orthogonal Tracy-Widom law (I. M. Johnstone [14]). For recent reviews we refer the reader to [4, 5, 6, 13, 27]. In these applications of Tracy-Widom distributions, one would like to control quantitatively the range of validity of the various limit laws. One therefore needs finite n correction to these limiting distributions. In a previous work [2] we initiated the study of this problem for the Gaussian Unitary Ensemble of n by n matrices (GUE_n). Following this work, we will derive the analogous result for the Gaussian Orthogonal Ensemble (GOE_n) in this paper. The derivation of the probability distribution function of the largest eigenvalue for Gaussian Symplectic Ensemble (GSE_n) is similar to the GOE_n case up to the parity of the size n of matrices in consideration. We will therefore mention at each step of the derivation the corresponding result without much explanations except when it is necessary. We seek a large n expansion of the probability distribution function of the largest eigenvalue in GOE_n and GSE_n similar to the following Edgeworth Expansion arising in probability in applications of the Central Limit Theorem.

If S_n is a sum of i.i.d. random variables X_j, each with mean μ and variance σ^2, then the distribution F_n of the normalized random variable $(S_n - n\mu)/(\sigma\sqrt{n})$ satisfies the Edgeworth expansion[1]

$$F_n(x) - \Phi(x) = \phi(x) \sum_{j=3}^{r} n^{-\frac{1}{2}j+1} R_j(x) + o(n^{-\frac{1}{2}r+1}) \tag{1.1}$$

uniformly in x. Here Φ is the standard normal distribution with density ϕ, and R_j are polynomials depending only on $\mathbb{E}(X_j^k)$ but not on n and r (or the underlying distribution of the X_j).

If we view the random matrix ensembles of n by n matrices in terms of the associated eigenvalues, then the Gaussian β-ensembles are probability spaces on n-tuples of random variables $\{\lambda_1, \ldots, \lambda_n\}$ (think of them as eigenvalues of a randomly chosen matrix from the ensemble.) The probability density that the eigenvalues lie in an infinitesimal intervals about the points $x_1, \ldots, x_n$ is

$$\mathbb{P}_{n\beta}(x_1, \cdots, x_n) \; = \mathrm{C}_{n\beta} \exp\left(-\frac{\beta}{2} \sum_1^n x_j^2 \right) \prod_{j<k} |x_j - x_k|^\beta, \tag{1.2}$$

with

$$-\infty < x_i < \infty, \quad \text{for } i = 1, \cdots, n. \tag{1.3}$$

Here $\mathrm{C}_{n\beta}$ is the normalizing constant such that the total integral over the x_i's is one. The cases $\beta = 1,\ 2,\ 4$ correspond to the GOE_n, GUE_n and GSE_n respectively. We denote the largest eigenvalue by λ_{Max}^β, and

$$F_{n,\beta}(t) = \mathbb{P}(\lambda_{\max}^\beta \le t) \tag{1.4}$$

the probability distribution function.
When $\beta = 2$, the harmonic oscillator wave functions

$$\varphi_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} H_k(x)\, e^{-x^2/2} \quad k = 0,\ 1,\ 2, \ldots$$

obtained by orthonormalizing the sequence $x^k e^{-x^2}$ (with $H_k(x)$ the Hermite polynomials of degree k) play an important role. We also have the Hermite kernel

$$K_{n,2}(x,y) = \sum_{k=0}^{n-1} \varphi_k(x)\varphi_k(y) = \sqrt{\frac{n}{2}} \quad \frac{\varphi_n(x)\varphi_{n-1}(y) - \varphi_n(y)\varphi_{n-1}(x)}{x-y}, \tag{1.5}$$

which is the kernel of an integral operator K_2 acting on $L^2(t,\infty)$, with resolvent

$$R_n(x,y;t) \; = \; (I - K_n)^{-1} K_n(x,y). \tag{1.6}$$

[1]We assume, of course, the moments $\mathbb{E}(X_j^k)$, $k = 3, \ldots, r$, exist; and as well, the condition $\lim_{|\zeta|\to\infty} \sup |\varphi(\zeta)| < \infty$ where φ is the characteristic function of X_j, see [7].

The product on the right is operator multiplication. We have the following representation of (1.2), (see for example, [16])

$$\mathbb{P}_{n2}(x_1, \cdots, x_n) \;=\; \det(K_n(x_i, x_j))_{1\le i\,j\,\le n}.$$

Following Tracy and Widom [24], we define

$$\varphi(x) \;=\; \left(\frac{n}{2}\right)^{\frac{1}{4}} \varphi_n(x), \qquad \psi(x) \;=\; \left(\frac{n}{2}\right)^{\frac{1}{4}} \varphi_{n-1}(x), \tag{1.7}$$

by ε the integral operator with kernel

$$\varepsilon_t(x) \;=\; \frac{1}{2}\mathrm{sgn}(x-t), \tag{1.8}$$

D the differentiation with respect to the independent variable,

$$Q_{n,i}(x;t) \;=\; (\,(I-K_n)^{-1}\,,\,x^i\varphi_n) \tag{1.9}$$

and

$$P_{n,i}(x;t) \;=\; (\,(I-K_n)^{-1}\,,\,x^i\varphi_{n-1}). \tag{1.10}$$

We introduce the following quantities

$$q_{n,i}(t) \;=\; Q_{n,i}(t;t), \quad p_{n,i}(t) \;=\; P_{n,i}(t;t) \tag{1.11}$$

$$u_{n,i}(t) \;=\; (Q_{n,i}, \varphi_n), \quad v_{n,i}(t) \;=\; (P_{n,i}, \varphi_n), \tag{1.12}$$

$$\tilde{v}_{n,i}(t) = (Q_{n,i}, \varphi_{n-1}), \quad \text{and} \quad w_{n,i}(t) \;=\; (P_{n,i}, \varphi_{n-1}). \tag{1.13}$$

Here $(\,\cdot\,,\cdot\,)$ denotes the inner product on $L^2(t,\infty)$. In our notation, the subscript without the n represents the scaled limit of that quantity when n goes to infinity, and we dropped the second subscript i when it is zero.
If Ai is Airy function, the kernel $K_{n,2}(x,y)$ then scales[2] to the Airy kernel

$$K_{\mathrm{Ai}}(X,Y) \;=\; \frac{\mathrm{Ai}(X)\;\mathrm{Ai}'(Y)\;-\;\mathrm{Ai}(Y)\;\mathrm{Ai}'(X)}{X-Y}. \tag{1.14}$$

Our conventions are as follows:

$$Q_i(x;s) \;=\; (\,(I-K_{\mathrm{Ai}})^{-1}\,,\,x^i\,\mathrm{Ai}), \quad Q_0(x;s) \;=\; Q(x;s), \tag{1.15}$$

$$P_i(x;s) \;=\; (\,(I-K_{\mathrm{Ai}})^{-1}\,,\,x^i\,\mathrm{Ai}'), \quad P_0(x;s) \;=\; P(x;s), \tag{1.16}$$

$$q_i(s) \;=\; Q_i(s;s), \quad q_0(s) = q(s), \quad p_i(s) \;=\; P_i(s;s), \quad p_0(s) = p(s), \tag{1.17}$$

$$u_i(s) \;=\; (Q_i, \mathrm{Ai}), \quad u_0(s) = u(s), \quad v_i(s) \;=\; (P_i, \mathrm{Ai}), \quad v_0(s) = v(s), \tag{1.18}$$

$$\tilde{v}_i(s) = (Q_i, \mathrm{Ai}'), \quad \tilde{v}_0(s) = \tilde{v}(s), \quad w_i(t) \;=\; (P_i, \mathrm{Ai}'), \quad \text{and} \quad w_0(s) = w(s). \tag{1.19}$$

[2]For the precise definition of this scaling, see the next section

Here $(\cdot\,,\cdot)$ denotes the inner product on $L^2(s,\infty)$ and $i=0,1,2$.
We also note that $q(s)$ is the solution to the Pailevé II equation $q''(s)=sq(s)+2q^3(s)$ with the boundary condition $q(s)\sim \mathrm{Ai}(s)$ as $s\to\infty$.
We use the subscript n for unscaled quantities only.

$$\mathcal{R}_{n,1}:=\int_{-\infty}^{t} R_n(x,t;t)\mathrm{d}x,\quad \mathcal{P}_{n,1}:=\int_{-\infty}^{t} P_n(x;t)\mathrm{d}x,\quad \mathcal{Q}_{n,1}:=\int_{-\infty}^{t} Q_n(x;t)\mathrm{d}x, \tag{1.20}$$

and

$$\mathcal{R}_{n,4}(t):=\int_{-\infty}^{\infty}\varepsilon_t(x)R_n(x,t;t)\mathrm{d}x,\quad \mathcal{P}_{n,4}(t):=\int_{-\infty}^{\infty}\varepsilon_t(x)P_n(x;t)\mathrm{d}x,$$
$$\mathcal{Q}_{n,4}(t):=\int_{-\infty}^{\infty}\varepsilon_t(x)Q_n(x;t)\mathrm{d}x. \tag{1.21}$$

The epsilon quantities are

$$Q_{n,\varepsilon}(x;t) \;=\; \big((I-K_n)^{-1}(x,y)\,,\,\varepsilon\varphi(y)\big),\quad q_{n,\varepsilon}(t)\;=\;Q_{n,\varepsilon}(t;t) \tag{1.22}$$

$$u_{n,\varepsilon}(t)\;=\;\big(Q_{n,\varepsilon}(x;t),\varphi(x)\big),\quad \tilde{v}_{n,\varepsilon}(t)\;=\;\big(Q_{n,\varepsilon}(x;t),\psi(x)\big), \tag{1.23}$$

where $(\cdot\,,\cdot)$ denotes the inner product on $L^2(t,\infty)$.

The GOE$_n$ and GSE$_n$ analogue of (1.28) in Theorem 1.1 bellow will follow from representations (equations (40) and (41) of [24].)

$$F_{n,1}(t)^2=F_{n,2}(t)\cdot\left(\big(1-\tilde{v}_{n,\varepsilon}(t)\big)\big(1-\frac{1}{2}\mathcal{R}_{n,1}(t)\big)-\frac{1}{2}\big(q_{n,\varepsilon}(t)-c_\varphi\big)\mathcal{P}_{n,1}(t)\right) \tag{1.24}$$

and

$$F_{n,4}(t/\sqrt{2})^2=F_{n,2}(t)\cdot\left(\big(1-\tilde{v}_{n,\varepsilon}(t)\big)\big(1+\frac{1}{2}\mathcal{R}_{n,4}(t)\big)+\frac{1}{2}q_{n,\varepsilon}(t)\,\mathcal{P}_{n,4}(t)\right). \tag{1.25}$$

Here we first derive a large n-expansion of $\mathcal{R}_{n,1}$, $\mathcal{P}_{n,1}$, $\mathcal{R}_{n,4}$, $\mathcal{P}_{n,4}$, $\tilde{v}_{n,\varepsilon}$, and $q_{n,\varepsilon}$ in terms of p_n and q_n, by solving the associated systems of differential equations. We then substitute the resulting expressions in (1.24) and (1.25). We will need the following result which gives the large n expansion of $F_{n,2}$.

Theorem 1.1. *[2] If we set*

$$t=(2(n+c))^{\frac{1}{2}}+2^{-\frac{1}{2}}n^{-\frac{1}{6}}\,s\quad\text{and} \tag{1.26}$$

$$E_{c,2}:=E_{c,2}(s)=2w_1-3u_2+(-20c^2+3)v_0+u_1v_0-u_0v_1+u_0v_0^2-u_0^2w_0. \tag{1.27}$$

Then as $n\to\infty$

$$F_{n,2}(t)=F_2(s)\left\{1+c\,u_0(s)\,n^{-\frac{1}{3}}-\frac{1}{20}E_{c,2}(s)\,n^{-\frac{2}{3}}\right\}+O(n^{-1}) \tag{1.28}$$

uniformly in s, and

$$F_2(s)\;=\;\lim_{n\to\infty}F_{n,2}(t)\;=\;\exp\left(-\int_s^\infty (x-s)q(x)^2\,dx\right) \tag{1.29}$$

is the Tracy-Widom distribution.

1.1 Statement of our results

To state our main result we need the following definitions

$$\alpha := \alpha(s) \;=\; \int_s^\infty q(x)\,u(x)\,dx, \tag{1.30}$$

$$\mu := \mu(s) \;=\; \int_s^\infty q(x)dx, \tag{1.31}$$

$$\nu := \nu(s) \;=\; \int_s^\infty p(x)dx \;=\; \alpha(s) \;-\; q(s), \tag{1.32}$$

$$\eta := \eta(s) = \frac{1}{20\sqrt{2}}\int_s^\infty \big(6qv + 3pu + 2p_2 + 2p_1 v + 2pv_1 - 2q_2 u - 2q_1 u_1 - 2qu_2\big)(x)\,dx - \frac{20c^2 q'(s) + 3p(s)}{20\sqrt{2}} \tag{1.33}$$

$$E_{c,1}(s) \;=\; -\frac{1}{20}E_{c,2}(s)\,e^{-\mu} - \frac{c\,\alpha}{2\mu^2} + \frac{c\,p}{2\mu} + \frac{(2c-1)\,\nu^2}{4\mu^2} + c\,u\left(c\,q\,e^{-\mu} - \frac{\nu}{2\mu}(1-e^{-\mu})\right) + e^{-2\mu}\left(\frac{\nu\,(\nu + 8c\,q)}{32\mu} - \frac{\eta}{4\sqrt{2}}\right) + e^{-\mu}\left(\frac{2\sqrt{2}\,c^2 q^2 - 3\,\eta}{4\sqrt{2}} + \frac{\nu^2 - 8(2c\,p + c^2 q^2) - 4c^2\alpha^2}{32\mu} - \frac{c^2 q^2}{8\mu^2} + \frac{2-\mu}{2\mu^2}\left(c\,q\,\alpha + \frac{1}{4}\nu^2 + (c^2 - c)q^2\right)\right) - \left(4c^2\alpha^2 + 3c^2 q^2 - \nu^2\right)\frac{\cosh(\mu)}{8\mu^2}. \tag{1.34}$$

Unfortunately, we did not find a simple representation of η and $E_{c,1}$. Nevertheless the quantities $\alpha(s)$, $\mu(s)$, $\nu(s)$, $\eta(s)$ and $E_{c,1}(s)$ are easy to compute numerically. For $E_{c,1}(s)$ and $\eta(s)$ we only need the recurrences relations defining $p_i(s)$ and $q_i(s)$ in term of $q(s)$ and $q'(s)$. We find the following representation of the large n probability distribution function for λ_{max} in GOE$_n$. The derivation of the analogous result for the GSE$_n$ follows exactly the same steps as the one given in this paper. Here is our main result.

Theorem 1.2. *We set*

$$t = (2(n+c))^{\frac{1}{2}} + 2^{-\frac{1}{2}} n^{-\frac{1}{6}}\, s. \tag{1.35}$$

Then as $n \to \infty$

$$F_{n,1}(t)^2 \;=\; F_2(s)\cdot\left\{e^{-\mu(s)} \;+\; \left[c\big(q(s)+u(s)\big)e^{-\mu(s)} - \frac{\nu(s)}{2\mu(s)}\big(1 - e^{-\mu(s)}\big)\right] n^{-\frac{1}{3}} \;+\; E_{c,1}(s) n^{-\frac{2}{3}}\right\} \;+\; O(n^{-1}) \tag{1.36}$$

uniformly for bounded s.

Note that unlike the GUE_n case where for $c = 0$ the $n^{-\frac{1}{3}}$ correction term vanishes as shown in equation (1.28), the $n^{-\frac{1}{3}}$ correction term does not vanish in the GOE_n no matter what the fine tuning constant c is.

In §2 we reproduce the derivation of (1.24) following Tracy and Widom in [24]. In §3 we solve the system of equations satisfied by the various functions on the right of (1.24) for our derivation of the Edgeworth expansion of the probability distribution of the largest eigenvalue in the GOE_n.

2 Derivation of $F_{n,\beta}$

We treat here the case n even. If we set

$$K_{n,1} \;=\; \begin{pmatrix} K_{n,2} + \psi\otimes\varepsilon\varphi & K_{n,2}\,D \;-\; \psi\otimes\varphi \\ \varepsilon\,K_{n,2} - \varepsilon + \varepsilon\,\psi\otimes\varepsilon\varphi & K_{n,2} + \varepsilon\psi\otimes\varphi \end{pmatrix} \tag{2.1}$$

then $F_{n,1}^2(t)$ is the Fredholm determinant of $K_{n,1}$ on the set $J = (t,\infty)$. If we denote by χ the multiplication by the function $\chi_J(x)$, then $F_{n,1}^2(t)$ is the Fredholm determinant of the integral operator with kernel[3]

$$\begin{aligned} K_{n,1} \;&=\; \chi_J(x)\begin{pmatrix} K_{n,2} + \psi\otimes\varepsilon\varphi & K_{n,2}\,D \;-\; \psi\otimes\varphi \\ \varepsilon\,K_{n,2} - \varepsilon + \varepsilon\,\psi\otimes\varepsilon\varphi & K_{n,2} + \varepsilon\psi\otimes\varphi \end{pmatrix}\chi_J(y) \\ &=\; \chi\begin{pmatrix} K_{n,2} + \psi\otimes\varepsilon\varphi & K_{n,2}\,D \;-\; \psi\otimes\varphi \\ \varepsilon\,K_{n,2} - \varepsilon + \varepsilon\,\psi\otimes\varepsilon\varphi & K_{n,2} + \varepsilon\psi\otimes\varphi \end{pmatrix}\chi \end{aligned} \tag{2.2}$$

on $\mathbb{R}$, see for example [24] equation (31). Using the following commutators,

$$[K_{n,2}, D] \;=\; \varphi\otimes\psi + \psi\otimes\varphi, \qquad [\varepsilon, K_{n,2}] \;=\; -\varepsilon\varphi\otimes\varepsilon\psi - \varepsilon\psi\otimes\varepsilon\varphi \tag{2.3}$$

(ψ and φ appear as a consequence of the Christophel Darboux formula applied to $K_{n,2}$,) we have

$$K_{n,2} + \psi\otimes\varphi = D\,\varepsilon K_{n,2} + D\,\varepsilon\psi\otimes\varepsilon\varphi = D(\varepsilon K_{n,2} + \varepsilon\psi\otimes\varepsilon\varphi) = D(K_{n,2}\,\varepsilon - \varepsilon\varphi\otimes\varepsilon\psi)$$

$$K_{n,2}\,D - \psi\otimes\varepsilon = D\,K_{n,2} + \varphi\otimes\psi = DK_2 + D\varepsilon\psi\otimes\varepsilon = D\,(K_{n,2} + \varepsilon\varphi\otimes\psi)$$

and

$$\varepsilon\,K_{n,2} - \varepsilon + \varepsilon\,\psi\otimes\varepsilon\varphi = K_{n,2}\,\varepsilon - \varepsilon - \varepsilon\,\varphi\otimes\varepsilon\psi$$

as $D\varepsilon \;=\; I$. Our kernel is now

$$K_{n,1} \;=\; \chi\begin{pmatrix} D(K_{n,2}\,\varepsilon - \varepsilon\varphi\otimes\varepsilon\psi) & D\,(K_{n,2} + \varepsilon\varphi\otimes\psi) \\ K_{n,2}\,\varepsilon - \varepsilon - \varepsilon\,\varphi\otimes\varepsilon\psi & K_{n,2} + \varepsilon\psi\otimes\varphi \end{pmatrix}\chi \tag{2.4}$$

$$= \begin{pmatrix} \chi D & 0 \\ 0 & \chi \end{pmatrix}\cdot\begin{pmatrix} (K_{n,2}\,\varepsilon - \varepsilon\varphi\otimes\varepsilon\psi)\chi & (K_{n,2} + \varepsilon\varphi\otimes\psi)\chi \\ (K_{n,2}\,\varepsilon - \varepsilon - \varepsilon\,\varphi\otimes\varepsilon\psi)\,\chi & (K_{n,2} + \varepsilon\psi\otimes\varphi)\,\chi \end{pmatrix} \tag{2.5}$$

[3] To simplify notation we kept the same notation for the integral operator as well as the kernel

Since $K_{n,1}$ is of the form AB, we can use the fact that $\det(I - \mathrm{AB}) = \det(I - \mathrm{BA})$ and deduce that the Fredholm determinant of $K_{n,1}$ is unchanged if instead we take $K_{n,1}$ to be

$$\begin{pmatrix} (K_{n,2}\,\varepsilon - \varepsilon\varphi \otimes \varepsilon\psi)\chi & (K_{n,2} + \varepsilon\varphi \otimes \psi)\chi \\ (K_{n,2}\,\varepsilon - \varepsilon - \varepsilon\,\varphi \otimes \varepsilon\psi)\,\chi & (K_{n,2} + \varepsilon\psi \otimes \varphi)\,\chi \end{pmatrix} \cdot \begin{pmatrix} \chi\, D & 0 \\ 0 & \chi \end{pmatrix} \tag{2.6}$$

$$= \begin{pmatrix} (K_{n,2}\,\varepsilon - \varepsilon\varphi \otimes \varepsilon\psi)\chi\, D & (K_{n,2} + \varepsilon\varphi \otimes \psi)\chi \\ (K_{n,2}\,\varepsilon - \varepsilon - \varepsilon\,\varphi \otimes \varepsilon\psi)\,\chi\, D & (K_{n,2} + \varepsilon\psi \otimes \varphi)\,\chi \end{pmatrix} \tag{2.7}$$

$$\det(I - K_{n,1}) \;=\; \det \begin{pmatrix} I - (K_{n,2}\,\varepsilon - \varepsilon\varphi \otimes \varepsilon\psi)\chi\, D & -(K_{n,2} + \varepsilon\varphi \otimes \psi)\chi \\ -(K_{n,2}\,\varepsilon - \varepsilon - \varepsilon\,\varphi \otimes \varepsilon\psi)\,\chi\, D & I - (K_{n,2} + \varepsilon\psi \otimes \varphi)\,\chi \end{pmatrix}. \tag{2.8}$$

Performing row and column operations on the matrix[4] does not change the Fredholm determinant. We first subtract row 1 from row 2, next we add column 2 to column 1 to have the following matrix

$$\begin{pmatrix} I - (K_{n,2}\,\varepsilon - \varepsilon\varphi \otimes \varepsilon\psi)\chi\, D - (K_{n,2} + \varepsilon\varphi \otimes \psi)\chi & -(K_{n,2} + \varepsilon\varphi \otimes \psi)\chi \\ \varepsilon\,\chi\, D & I \end{pmatrix}. \tag{2.9}$$

Right-multiply column 2 by $-\varepsilon\,\chi\, D$ and add it to column 1, and multiply row 2 by $(K_{n,2} + \varepsilon\varphi \otimes \psi)\chi$ and add it to row 1 to have

$$\begin{pmatrix} I - (K_{n,2}\,\varepsilon - \varepsilon\varphi \otimes \varepsilon\psi)\chi\, D - (K_{n,2} + \varepsilon\varphi \otimes \psi)\chi + (K_{n,2} + \varepsilon\varphi \otimes \psi)\chi\,\varepsilon\,\chi\, D & 0 \\ 0 & I \end{pmatrix}. \tag{2.10}$$

We therefore have,

$$\det(I - K_{n,1}) \;=\; \det\big(I - (K_{n,2}\,\varepsilon - \varepsilon\varphi \otimes \varepsilon\psi)\chi\, D + (K_{n,2} + \varepsilon\varphi \otimes \psi)\chi\,\big(\varepsilon\,\chi\, D - I\big)\big) \tag{2.11}$$

$$=\; \det\left(I - K_{n,2}\chi - K_{n,2}(I-\chi)\varepsilon\chi\, D - (\varepsilon\varphi \otimes \psi)(\chi - \chi\varepsilon\chi D) - \varepsilon\varphi \otimes \psi\varepsilon\chi D\right) \tag{2.12}$$

We used the fact that ε is antisymmetric to have

$$\varepsilon\varphi \otimes \varepsilon\psi\,\chi\, D = \varepsilon\varphi \otimes \psi\varepsilon^t\,\chi\, D = -\varepsilon\varphi \otimes \psi\varepsilon\,\chi\, D,$$

and if we note that χ is multiplication, then the determinant is

$$=\; \det\left(I - K_{n,2}\chi - K_{n,2}(I-\chi)\varepsilon\chi\, D - \varepsilon\varphi \otimes \psi(1-\chi)\varepsilon\chi D - \varepsilon\varphi \otimes \chi\psi\right).$$

Next we factor out $I - K_{n,2}$ and note that $(I - K_{n,2})^{-1} = I + R_{n,2}$, where $R_{n,2}$ was defined as the resolvent of $K_{n,2}$, and $(I - K_{n,2})^{-1}\varepsilon\varphi = Q_{n,\varepsilon}$. We are interested on the determinant of the following operator

$$\big(I - K_{n,2}\chi\big)\big(I - (K_{n,2} + R_{n,2}K_{n,2})(I-\chi)\varepsilon\chi\, D - Q_{n,\varepsilon}\otimes\psi(1-\chi)\varepsilon\chi D - Q_{n,\varepsilon}\otimes\chi\psi\big). \tag{2.13}$$

We have a large n-expansion of $\det(I - K_{n,2}\chi) = F_{n,2}$ from the author work in GUE_n see [2] or [3]. We will therefore focus our attention on the second factor of

[4]This does not change the determinant, for more details see [24]

(2.13). We will represent this factor in the form $(I-\sum_{j=1}^{k}\alpha_j\otimes\beta_j)$ and use the well known formula $\det(I-\sum_{j=1}^{k}\alpha_j\otimes\beta_j)=\det\big(\delta_{i,j}-(\alpha_i,\beta_j)\big)_{i,j=1,\ldots,k}$ to expand the Fredholm determinant. First we need to find a representation of $\varepsilon\chi\, D$ as a finite rank operator. To this end we introduce in this section the following notation,

$$\varepsilon_k(x)=\varepsilon(x-a_k),\quad R_k(x)=R_{n,2}(x,a_k),\quad \delta_k(x)=\delta(x-a_k),\quad a_1=t,\quad \text{and}\quad a_2=\infty.$$

With the new notation $J=(t,\infty)=(a_1,a_2)$, and the commutator

$$[\chi\ \ D]=-\delta_1\otimes\delta_1+\delta_2\otimes\delta_2,$$

gives

$$\varepsilon[\chi\ \ D]=-\varepsilon_1\otimes\delta_1+\varepsilon_2\otimes\delta_2.$$

Next we use the identity $\varepsilon\, D=I$ to have

$$(I-\chi)\varepsilon\,\chi\, D=(I-\chi)\varepsilon\,[\chi\ \ D]=-(I-\chi)\varepsilon_1\otimes\delta_1+(I-\chi)\varepsilon_2\otimes\delta_2, \tag{2.14}$$

and the representation

$$(K_{n,2}+R_{n,2}\,K_{n,2})(I-\chi)\varepsilon\,\chi\, D=\sum_{k=1,2}(-1)^k(K_{n,2}+R_{n,2}\,K_{n,2})(I-\chi)\varepsilon_k\otimes\delta_k. \tag{2.15}$$

We substitute (2.14) and (2.15) in the second factor of (2.13) and have,

$$I-\sum_{k=1,2}(-1)^k(K_{n,2}+R_{n,2}K_{n,2})(I-\chi)\varepsilon_k\otimes\delta_k-\sum_{k=1,2}(-1)^kQ_{n,\varepsilon}\otimes\psi\cdot(1-\chi)\varepsilon_k\otimes\delta_k-Q_{n,\varepsilon}\otimes\chi\psi. \tag{2.16}$$

The dot in this formula represent operator multiplication. In this case we just multiply the kernels using the formula $(\alpha\otimes\beta)(\gamma\otimes\delta)=(\beta,\gamma)\alpha\otimes\delta$, to have the following form of (2.16)

$$I-\sum_{k=1,2}(-1)^k(K_{n,2}+R_{n,2}K_{n,2})(I-\chi)\varepsilon_k\otimes\delta_k-\sum_{k=1,2}(-1)^k\big(\psi,(I-\chi)\varepsilon_k\big)Q_{n,\varepsilon}\otimes\delta_k-Q_{n,\varepsilon}\otimes\chi\psi. \tag{2.17}$$

We have

$$\varepsilon_2=-\frac{1}{2},\quad (I-\chi)\varepsilon_1=(I-\chi)\varepsilon_2=-\frac{1}{2},\quad \text{and}\quad R_2=R_{n,2}(x,\infty)=0.$$

If we substitute these value in (2.17), it then becomes,

$$I-Q_{n,\varepsilon}\otimes\chi\psi-\frac{1}{2}\big[(k_{n,2}+R_{n,2}\,K_{n,2})(I-\chi)+\big(\psi,(I-\chi)\big)Q_{n,\varepsilon}\big]\otimes(\delta_1-\delta_2). \tag{2.18}$$

This operator is of the desired form

$$I-\sum_{k=1,2}\alpha_k\otimes\beta_k$$

with

$$\alpha_1 = Q_{n,\varepsilon},\ \alpha_2 = \frac{1}{2}\big[(k_{n,2}+R_{n,2}\,K_{n,2})(I-\chi)+a_1 Q_{n,\varepsilon}\big],\ \beta_1 = \chi\psi,\ \beta_2 = \delta_1 - \delta_2, \tag{2.19}$$

and

$$a_1 = \big(\psi, (I-\chi)\big).$$

The corresponding inner product are;

$$(\alpha_1, \beta_1) = \tilde{v}_{n,\varepsilon}, \quad (\alpha_1, \beta_2) = q_{n,\varepsilon} + c_\varphi \tag{2.20}$$

with

$$c_\varphi = \varepsilon\,\varphi(\infty) = \frac{1}{2}\int_{-\infty}^{\infty}\varphi(x)\,dx \quad c_\psi = \varepsilon\,\psi(\infty) = \frac{1}{2}\int_{-\infty}^{\infty}\psi(x)\,dx, \tag{2.21}$$

and for n even

$$c_\varphi = (\pi\,n)^{1/4}2^{-3/4-n/2}\frac{(n!)^{1/2}}{(n/2)!}, \tag{2.22}$$

and

$$(\alpha_2, \beta_1) = \frac{1}{2}(\mathcal{P}_{n,1} - a_1 + a_1\tilde{v}_{n,\varepsilon}), \quad (\alpha_2, \beta_2) = \frac{1}{2}(\mathcal{R}_{n,1} + a_1 q_{n,\varepsilon} - a_1 c_\varphi). \tag{2.23}$$

The determinant of (2.17) is therefore

$$(1-\tilde{v}_{n,\varepsilon})(1-\frac{1}{2}\mathcal{R}_{n,1}) - \frac{1}{2}(q_{n,\varepsilon} - c_\varphi)\mathcal{P}_{n,1}. \tag{2.24}$$

In a similar way we obtain the second factor on the right side of (1.25) for the GSE$_n$ case

$$\big(1-\tilde{v}_{n,\varepsilon}\big)\big(1+\frac{1}{2}\mathcal{R}_{n,4}\big) + \frac{1}{2}q_{n,\varepsilon}(t)\,\mathcal{P}_{n,4}(t). \tag{2.25}$$

We will derive differential equations involving the various terms in equation (2.24) and solutions in terms of q_n and p_n. Then used known asymptotic of q_n and p_n for the derivation of a large n expansion of $F_{n,1}$.

3 Differential Equations

3.1 System of Differential Equations

This section will be devoted to finding expressions of

$$\mathcal{R}_{n,1}(t), \quad \mathcal{P}_{n,1}(t), \quad q_{n,\varepsilon}(t), \quad \text{and} \quad \tilde{v}_{n,\varepsilon}(t),$$

and show how to obtain the corresponding quantities

$$\mathcal{R}_{n,4}(t), \quad \text{and} \quad \mathcal{P}_{n,4}(t)$$

for the GSE$_n$.

To solve the associated system of differential equations it is convenient to introduce the following quantities

$$\mathcal{Q}_{n,1}(t), \quad u_{n,\varepsilon}(t),$$

and

$$\rho_{n,2}(x,y) \quad \text{the kernel of } \ (I - K_{n,2}\chi)^{-1}. \tag{3.1}$$

We have

$$\rho_{n,2}(x,y) \ = \ \delta(x-y) \ + \ R_{n,2}(x,y), \tag{3.2}$$

and

$$\frac{d}{dt}\mathcal{R}_{n,1}(t) = \frac{d}{dt}\int_{-\infty}^{t} R_{n,2}(x,t)\,dx = R_{n,2}(t,t) + \int_{-\infty}^{t} \frac{d}{dt}R_{n,2}(x,t)\,dx. \tag{3.3}$$

Formula (45) of [24] gives

$$\frac{d}{dt}R_{n,2}(x,t) = -\frac{d}{dx}R_{n,2}(x,t) - p_n(t)Q_n(x;t) - q_n P_n(x;t), \tag{3.4}$$

and we find that

$$\mathcal{R}_{n,1}^{'}(t) \ = \ \frac{d}{dt}\mathcal{R}_{n,1}(t) = -p_n(t)\,\mathcal{Q}_{n,1}(t) - q_n(t)\,\mathcal{P}_{n,1}(t). \tag{3.5}$$

We also have

$$\mathcal{Q}_{n,1}^{'}(t) \ = \ \frac{d}{dt}\mathcal{Q}_{n,1}(t) \ = \ \frac{d}{dt}\int_{-\infty}^{t} Q_n(x;t)\,dx \ = \ q_n(t)(1-\mathcal{R}_{n,1}(t)), \tag{3.6}$$

and

$$\mathcal{P}_{n,1}^{'}(t) \ = \ \frac{d}{dt}\mathcal{P}_{n,1}(t) \ = \ \frac{d}{dt}\int_{-\infty}^{t} P_n(x;t)\,dx \ = \ p_n(t)(1-\mathcal{R}_{n,1}(t)), \tag{3.7}$$

where we used

$$\frac{d}{dt}Q_n(x;t) = -q_n(t)\,R_{n,2}(x,t), \quad \text{and} \quad \frac{d}{dt}P_n(x;t) = -p_n(t)\,R_{n,2}(x,t). \tag{3.8}$$

The other derivatives are

$$\frac{d}{dt}u_{n,\varepsilon}(t) = \frac{d}{dt}\int_{t}^{\infty} Q_n(x,t)\varepsilon\,\varphi(x)\,dx = -q_n(t)\varepsilon\varphi(t) + \int_{t}^{\infty} \frac{d}{dt}Q_n(x,t)\varepsilon\,\varphi(x)\,dx \tag{3.9}$$

$$= -q_n(t)\big(\varepsilon\varphi(t) + \int_{t}^{\infty} \frac{d}{dt}R_{n,2}(x,t)\varepsilon\,\varphi(x)\,dx\big) = -q_n(t)\int_{t}^{\infty} \rho_{n,2}(x,t)\varepsilon\,\varphi(x)\,dx, \tag{3.10}$$

and therefore

$$u_{n,\varepsilon}^{'}(t) \ = \ -q_n(t)\,q_{n,\varepsilon}(t). \tag{3.11}$$

Similarly

$$\frac{d}{dt}\tilde{v}_{n,\varepsilon}(t) = \frac{d}{dt}\int_{t}^{\infty} P_n(x,t)\varepsilon\,\varphi(x)\,dx = -p_n(t)\varepsilon\varphi(t) + \int_{t}^{\infty} \frac{d}{dt}P_n(x,t)\varepsilon\,\varphi(x)\,dx, \tag{3.12}$$

or

$$\tilde{v}^{'}_{n,\varepsilon}(t) \;\;=\;\; -p_n(t)\, q_{n,\varepsilon}(t). \tag{3.13}$$

The last of these is

$$\frac{d}{dt}q_{n,\varepsilon}(t) = \frac{d}{dt}\int \rho_{n,2}(t,y)\,\varepsilon\varphi(y)\,dy \tag{3.14}$$

$$= -\int \frac{\partial}{\partial y}\rho_{n,2}(t,y)\,\varepsilon\varphi(y)dy - q_n(t)\big(\chi P_n(y;t),\varepsilon\varphi(y)\big) - p_n(t)\big(\chi Q_n(y;t),\varepsilon\varphi(y)\big). \tag{3.15}$$

Integration by parts together with the boundary conditions and $D\varepsilon = I$ gives

$$-\int \frac{\partial}{\partial y}\rho_{n,2}(t,y)\,\varepsilon\varphi(y)dy = \int \rho_{n,2}(t,y)\,\chi\varepsilon\varphi(y)\,dy = q_n(t), \tag{3.16}$$

which in turn gives

$$q^{'}_{n,\varepsilon}(t) = q_n(t) - \tilde{v}_{n,\varepsilon}(t)q_n(t) - u_{n,\varepsilon}(t)q_n(t). \tag{3.17}$$

The boundary conditions at $t = \infty$ for these function are,

$$\mathcal{R}_{n,1}(\infty) = 0, \quad \mathcal{Q}_{n,1}(\infty) = 2c_\varphi \quad \text{and} \quad \mathcal{P}_{n,1}(\infty) = 2c_\psi = 0 \quad \text{as } n \text{ is even.} \tag{3.18}$$

and

$$\tilde{v}_{n,\varepsilon}(\infty) = 0, \quad u_{n,\varepsilon}(\infty) = 0, \quad \text{and} \quad q_{n,\varepsilon}(\infty) = c_\varphi. \tag{3.19}$$

The associated systems of equations are;

$$\left\{ \begin{array}{lcl} q^{'}_{n,\varepsilon}(t) & = & q_n(t)\big(1-\tilde{v}_{n,\varepsilon}(t)\big) - p_n(t)\,u_{n,\varepsilon}(t); \\ \big(1-\tilde{v}_{n,\varepsilon}\big)^{'}(t) & = & p_n(t)\,q_{n,\varepsilon}(t); \\ u^{'}_{n,\varepsilon}(t) & = & -q_n(t)\,q_{n,\varepsilon}(t). \end{array} \right. \tag{3.20}$$

and

$$\left\{ \begin{array}{lcl} \big(1-\mathcal{R}_{n,1}\big)^{'}(t) & = & p_n(t)\,\mathcal{Q}_{n,1}(t) + q_n(t)\,\mathcal{P}_{n,1}(t); \\ \mathcal{Q}^{'}_{n,1}(t) & = & q_n(t)(1-\mathcal{R}_{n,1}(t)); \\ \mathcal{P}^{'}_{n,1}(t) & = & p_n(t)(1-\mathcal{R}_{n,1}(t)). \end{array} \right. \tag{3.21}$$

3.2 Asymptotic solutions

We will define in this subsection only

$$V_{n,\varepsilon}(t) = 1 - \tilde{v}_{n,\varepsilon}(t), \quad \text{and} \quad \tilde{\mathcal{R}}_{n,1}(t) = 1 - \mathcal{R}_{n,1}(t). \tag{3.22}$$

With this notation, system (3.20) is

$$\frac{d}{dt}\begin{pmatrix} u_{n,\varepsilon}(t) \\ V_{n,\varepsilon}(t) \\ q_{n,\varepsilon}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -q_n(t) \\ 0 & 0 & p_n(t) \\ -p_n(t) & q_n(t) & 0 \end{pmatrix} \cdot \begin{pmatrix} u_{n,\varepsilon}(t) \\ V_{n,\varepsilon}(t) \\ q_{n,\varepsilon}(t) \end{pmatrix} \tag{3.23}$$

and (3.21) is

$$\frac{d}{dt}\begin{pmatrix}\mathcal{Q}_{n,1}(t)\\ \mathcal{P}_{n,1}(t)\\ \tilde{\mathcal{R}}_{n,1}(t)\end{pmatrix}=\begin{pmatrix}0 & 0 & q_n(t)\\ 0 & 0 & p_n(t)\\ p_n(t) & q_n(t) & 0\end{pmatrix}\cdot\begin{pmatrix}\mathcal{Q}_{n,1}(t)\\ \mathcal{P}_{n,1}(t)\\ \tilde{\mathcal{R}}_{n,1}(t)\end{pmatrix}. \tag{3.24}$$

If we let

$$X^t=\big(u_{n,\varepsilon}(t),V_{n,\varepsilon}(t),q_{n,\varepsilon}(t)\big)\quad\text{and}\quad Y^t=\big(\mathcal{Q}_{n,1}(t),\mathcal{P}_{n,1}(t),\tilde{\mathcal{R}}_{n,1}(t)\big),$$

then (3.20) and (3.21) have the following representations

$$X^{'}(t)\;=\;A(t)\,X(t)\quad\text{and}\quad Y^{'}(t)\;=\;B(t)\,Y(t) \tag{3.25}$$

with

$$X^t(\infty)\;=\;(0,1,c_\varphi),\quad\text{and}\quad Y^t(\infty)=(2c_\varphi,0,1). \tag{3.26}$$

We note that $A(t)$ is continuous for t bounded away from $-\infty$. We need to show that our matrix $A(t)$ is bounded as an operator on $L^1(t,\infty)\otimes L^1(t,\infty)\otimes L^1(t,\infty)$ for this end we will use the Max norm. The entries of $A(t)$ are $\pm q_n(x)$ and $\pm p_n(x)$.

$$\int_t^\infty |q_n(x)|\,dx=\frac{1}{\sqrt{2}}\int_s^\infty\left|q(x)+f(x)n^{-\frac{1}{3}}\right|dx=M_1\quad\text{with}\quad M_1\;<\;\infty \tag{3.27}$$

We made use of the following change of variables together with formula[5](2.29) of [3]

$$x=\sqrt{2n}+\frac{X}{\sqrt{2}n^{\frac{1}{6}}},\quad\text{and}\quad t=\sqrt{2n}+\frac{s}{\sqrt{2}n^{\frac{1}{6}}}, \tag{3.28}$$

the fact that the asymptotics for p at infinity can be obtained from the following representation $p=q^{'}+uq$ where $u(\infty)=0$ or that $u(x)$ is bounded for x away from minus infinity. We also assumed without lost of generalities for this section that $p(x)\sim\text{Const}\cdot x^{1/4}e^{-\frac{2}{3}x^{\frac{3}{2}}}$ as $x\to\infty$ which is a consequence of the asymptotics $q(x)\sim\frac{1}{2\sqrt{\pi}x^{1/4}}e^{-\frac{2}{3}x^{\frac{3}{2}}}$ as $x\to\infty$. We also remarked that the scaled value of q_n in (2.29) of [3] is represented in terms of finite combinations of bounded functions (the u_i's, v_i's, w_i's ,) x^i, $i=0,1,2$, with p and q. So again we assumed that the scaled value of $q_n(x)$ was of order $n^{\frac{1}{6}}x^2e^{-\frac{2}{3}x^{\frac{3}{2}}}$ as $x\to\infty$. A similar argument hold for p_n, here we use formula (2.30) of [3] instead.

$$\int_t^\infty |p_n(x)|\,dx=\frac{1}{\sqrt{2}}\int_s^\infty\left|q(x)+g(x)n^{-\frac{1}{3}}\right|dx=M_2\quad\text{with}\quad M_2\;<\;\infty. \tag{3.29}$$

Note that $||A||_{Max}$ and $||B||_{Max}$ are at most $2M_1+2M_2$. The fundamental local existence of solution for linear Ordinary Differential Equation says that equations (3.25) have solutions in (a,∞) with a bounded away from infinity given by

$$X(t)=\exp\left(-\int_t^\infty A(x)dx\right)\cdot X(\infty),\quad\text{and}\quad Y(t)=\exp\left(-\int_t^\infty B(x)dx\right)\cdot Y(\infty). \tag{3.30}$$

[5]We set the constant c in (2.29) to zero, and use the known asymptotic of q(x) at infinity to deduce the existance of the integral.

This solution is convenient for the large n expansion of the probability distribution since it allows us to give a series expansion of the solutions of our solution in terms of q_n and p_n. The other advantage is the built in symmetries in matrices A and B. These symmetries make the computation of the matrix exponential very easy. We will start with the first system $X(t) = \exp\left(-\int_t^\infty A(x)dx\right) \cdot X(\infty)$. We set

$$\exp\left(-\int_t^\infty A(x)\,dx\right) = \exp\left\{\begin{pmatrix} 0 & 0 & \int_t^\infty q_n(x)\,dx \\ 0 & 0 & -\int_t^\infty p_n(x)\,dx \\ \int_t^\infty p_n(x)dx & -\int_t^\infty q_n(x)\,dx & 0 \end{pmatrix}\right\}$$
$$= \exp(M). \tag{3.31}$$

M is of the form

$$M = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & -b \\ b & -a & 0 \end{pmatrix} \tag{3.32}$$

with

$$\exp(M) = \begin{pmatrix} 1+\sum_{k\geq 1}\frac{2^{k-1}a^k b^k}{(2k)!} & -\sum_{k\geq 1}\frac{2^{k-1}a^{k+1}b^{k-1}}{(2k)!} & \sum_{k\geq 0}\frac{2^k a^{k+1}b^k}{(2k+1)!} \\ -\sum_{k\geq 1}\frac{2^{k-1}a^{k-1}b^{k+1}}{(2k)!} & 1+\sum_{k\geq 1}\frac{2^{k-1}a^k b^k}{(2k)!} & -\sum_{k\geq 0}\frac{2^k a^k b^{k+1}}{(2k+1)!} \\ \sum_{k\geq 0}\frac{2^k a^k b^{k+1}}{(2k+1)!} & -\sum_{k\geq 0}\frac{2^k a^{k+1} b^k}{(2k+1)!} & 1+\sum_{k\geq 1}\frac{2^k a^k b^k}{(2k)!} \end{pmatrix}$$
$$= \begin{pmatrix} \exp(M)_{11} & \exp(M)_{12} & \exp(M)_{13} \\ \exp(M)_{21} & \exp(M)_{22} & \exp(M)_{23} \\ \exp(M)_{31} & \exp(M)_{32} & \exp(M)_{33} \end{pmatrix} \tag{3.33}$$

We have

$$u_{n,\varepsilon}(t) = \exp(M)_{12} + c_\varphi \exp(M)_{13}, \tag{3.34}$$
$$V_{n,\varepsilon}(t) = \exp(M)_{22} + c_\varphi \exp(M)_{23}, \tag{3.35}$$

and

$$q_{n,\varepsilon}(t) = \exp(M)_{32} + c_\varphi \exp(M)_{33}. \tag{3.36}$$

3.2.1 Scaling

At this point we scale the functions involved in (3.33) in terms of n at the point corresponding to the expected value of the largest eigenvalue. If we set

$$t = \tau(s) = \sqrt{2(n+c)} + \frac{s}{2^{\frac{1}{2}}n^{\frac{1}{6}}}, \tag{3.37}$$

then equations (2.29) and (2.30) of [3] are

$$q_n(\tau(s)) = Q_n(\tau(s);\tau(s)) = n^{\frac{1}{6}}\left(q(s) + \left[\frac{2c-1}{2}p(s) - cq(s)u(s)\right]n^{\frac{1}{3}}\right.$$

$$+\Big[(10c^2-10c+\frac{3}{2})q_1(s)+p_2(s)+(-30c^2+10c+\frac{3}{2})q(s)v(s)$$
$$+p_1(s)v(s)+p(s)v_1(s)-q_2(s)u(s)-q_1(s)u_1(s)-q(s)u_2(s)$$
$$+\ (-10c^2+\frac{3}{2})p(s)u(s)+20c^2q(s)u^2(s)\Big]\,\frac{n^{-\frac{2}{3}}}{20}+O(n^{-1})e_q(s)\Big), \tag{3.38}$$

and

$$p_n(\tau(s))\ =\ P_n(\tau(s);\tau(s))=n^{\frac{1}{6}}\Big(q(s)+\Big[\frac{2c+1}{2}p(s)-cq(s)u(s)\Big]\,n^{\frac{1}{3}}$$
$$+\Big[(10c^2+10c+\frac{3}{2})q_1(s)+p_2(s)+(-30c^2-10c+\frac{3}{2})q(s)v(s)$$
$$+p_1(s)v(s)+p(s)v_1(s)-q_2(s)u(s)-q_1(s)u_1(s)-q(s)u_2(s)$$
$$+\ (-10c^2+\frac{3}{2})p(s)u(s)+20c^2q(s)u^2(s)\Big]\,\frac{n^{-\frac{2}{3}}}{20}+O(n^{-1})e_p(s)\Big). \tag{3.39}$$

If we change the variable in a and b by setting[6] $x:=\tau(x)$, we obtain

$$a=\int_t^\infty q_n(x)\,dx\ =\frac{1}{\sqrt{2}}\int_s^\infty\Big(q(x)+\Big[\frac{2c-1}{2}p(x)-cq(x)u(x)\Big]\,n^{\frac{1}{3}}$$
$$+\Big[(10c^2-10c+\frac{3}{2})q_1(x)+p_2(x)+(-30c^2+10c+\frac{3}{2})q(x)v(x)$$
$$+p_1(x)v(x)+p(x)v_1(x)-q_2(x)u(x)-q_1(x)u_1(x)-q(x)u_2(x)$$
$$+\ (-10c^2+\frac{3}{2})p(x)u(x)+20c^2q(x)u^2(x)\Big]\,\frac{n^{-\frac{2}{3}}}{20}+O(n^{-1})e_q(x)\Big)\ dx, \tag{3.40}$$
$$=a_0(s)+a_1(s)n^{-1/3}+a_2(s)n^{-2/3}+a_3(s)n^{-1}$$

and

$$b=\int_t^\infty p_n(x)\,dx\ =\frac{1}{\sqrt{2}}\int_s^\infty\Big(q(x)+\Big[\frac{2c+1}{2}p(x)-cq(x)u(x)\Big]\,n^{\frac{1}{3}}$$
$$+\Big[(10c^2+10c+\frac{3}{2})q_1(x)+p_2(x)+(-30c^2-10c+\frac{3}{2})q(x)v(x)$$
$$+p_1(x)v(x)+p(x)v_1(x)-q_2(x)u(x)-q_1(x)u_1(x)-q(x)u_2(x)$$
$$+\ (-10c^2+\frac{3}{2})p(x)u(x)+20c^2q(x)u^2(x)\Big]\,\frac{n^{-\frac{2}{3}}}{20}+O(n^{-1})e_p(x)\Big)\ dx \tag{3.41}$$
$$=a_0(s)+b_1(s)n^{-1/3}+b_2(s)n^{-2/3}+b_3(s)n^{-1}.$$

[6]We use the same letter in both sides in the change here to simplify notation.

We next focus on the following expression

$$a^kb^k = (ab)^k = (a_0^2 + a_0(a_1+b_1)n^{-\frac{1}{3}} + (a_0(b_2+a_2)+a_1b_1)n^{-\frac{2}{3}} + Dn^{-1})^k.$$

An expansion of this expression is

$$a^kb^k = (ab)^k = a_0^{2k} + ka_0^{2k-2}\left(a_0(a_1+b_1)n^{-\frac{1}{3}} + (a_0(b_2+a_2)+a_1b_1)n^{-\frac{2}{3}} + Dn^{-1}\right) +$$
$$\frac{k(k-1)}{2}a_0^{2k-4}\left(a_0(a_1+b_1)n^{-\frac{1}{3}} + (a_0(b_2+a_2)+a_1b_1)n^{-\frac{2}{3}} + Dn^{-1}\right)^2 +$$
$$\sum_{i=3}^{k} a_0^{2k-2i}\binom{k}{i}\left(a_0(a_1+b_1)n^{-\frac{1}{3}} + (a_0(b_2+a_2)+a_1b_1)n^{-\frac{2}{3}} + Dn^{-1}\right)^i.$$

If we note that for $i \geq 3$

$$\left(a_0(a_1+b_1)n^{-\frac{1}{3}} + (a_0(b_2+a_2)+a_1b_1)n^{-\frac{2}{3}} + Dn^{-1}\right)^i = O(n^{-1}),$$

then the sum in this last term can be represented as

$$\sum_{i=3}^{k} a_0^{2k-2i}\binom{k}{i} O(n^{-1}) = \left(\sum_{i=3}^{k} a_0^{2k-2i}\binom{k}{i} + a_0^{2k} + ka_0^{2k-2} + \frac{k(k-1)}{2}a_0^{2k-4} - a_0^{2k} - ka_0^{2k-2}\right.$$
$$\left.-\frac{k(k-1)}{2}a_0^{2k-4}\right) O(n^{-1}) = \left((a_0^2+1)^k - a_0^{2k} - ka_0^{2k-2} - \frac{k(k-1)}{2}a_0^{2k-4}\right) O(n^{-1}).$$

We have

$$\frac{ka_0^{2k-2}}{(2k)!}\left(a_0(a_1+b_1)n^{-\frac{1}{3}} + (a_0(b_2+a_2)+a_1b_1)n^{-\frac{2}{3}} + Dn^{-1}\right) = \tag{3.42}$$
$$\frac{a_0^{2k-1}}{2(2k-1)!}\left((a_1+b_1)n^{-\frac{1}{3}} + (a_2+b_2)n^{-\frac{2}{3}}\right) + \frac{a_0^{2k-2}}{2(2k-1)!}a_1b_1n^{-\frac{2}{3}} + \frac{a_0^{2k-2}}{2(2k-1)!}Dn^{-1},$$

and for $k \geq 2$

$$\frac{k(k-1)}{2(2k)!}a_0^{2k-4}\left(a_0(a_1+b_1)n^{-\frac{1}{3}} + (a_0(b_2+a_2)+a_1b_1)n^{-\frac{2}{3}} + Dn^{-1}\right)^2 = \tag{3.43}$$
$$\left(\frac{a_0^{2k-2}}{8(2k-2)!} - \frac{a_0^{2k-2}}{8(2k-1)!}\right)\left((a_1+b_1)^2n^{-\frac{2}{3}} + O(n^{-1})\right).$$

We have at this stage,

$$1+\sum_{k\geq 1}\frac{2^{k-1}a^kb^k}{(2k)!} = 1+\sum_{k\geq 1}\frac{2^{k-1}a_0^{2k}}{(2k)!} + \sum_{k\geq 1}\frac{2^{k-1}a_0^{2k-1}}{2(2k-1)!}(a_1+b_1)n^{-\frac{1}{3}} + \left[\sum_{k\geq 1}\frac{2^{k-1}a_0^{2k-1}}{2(2k-1)!}(a_2+b_2)\right.$$
$$\left. + \sum_{k\geq 1}\frac{2^{k-1}a_0^{2k-2}}{2(2k-1)!}a_1b_1 + \left(\sum_{k\geq 2}\frac{2^{k-1}a_0^{2k-2}}{8(2k-2)!} - \sum_{k\geq 2}\frac{2^{k-1}a_0^{2k-2}}{8(2k-1)!}\right)(a_1+b_1)^2\right] n^{-\frac{2}{3}} +$$

$$\sum_{k\geq 1}\frac{2^{k-1}(a_0^2+1)^k}{(2k)!}\,O(n^{-1}) \tag{3.44}$$

$$= 1+\frac{1}{2}\sum_{k\geq 1}\frac{(\sqrt{2}a_0)^{2k}}{(2k)!}+\frac{(a_1+b_1)}{2\sqrt{2}}\sum_{k\geq 1}\frac{(\sqrt{2}a_0)^{2k-1}}{(2k-1)!}\,n^{-\frac{1}{3}}+\Bigg[\frac{(a_2+b_2)}{2\sqrt{2}}\sum_{k\geq 1}\frac{(\sqrt{2}a_0)^{2k-1}}{(2k-1)!}+$$

$$\frac{a_1b_1}{2\sqrt{2}a_0}\sum_{k\geq 1}\frac{(\sqrt{2}a_0)^{2k-1}}{(2k-1)!}+\left(\sum_{k\geq 1}\frac{(\sqrt{2}a_0)^{2k}}{8(2k)!}-\sum_{k\geq 1}\frac{(\sqrt{2}a_0)^{2k}}{8(2k+1)!}\right)(a_1+b_1)^2\Bigg]\,n^{-\frac{2}{3}}+$$

$$\sum_{k\geq 1}\frac{(2a_0^2+2)^k}{(2k)!}\,O(n^{-1}) \tag{3.45}$$

$$= \frac{1}{2}\left(1+\cosh(\sqrt{2}a_0)\right)+\frac{(a_1+b_1)}{2\sqrt{2}}\sinh(\sqrt{2}a_0)\,n^{-\frac{1}{3}}+\Bigg[\frac{(a_2+b_2)}{2\sqrt{2}}\sinh(\sqrt{2}a_0)+$$

$$\frac{a_1b_1}{2\sqrt{2}a_0}\sinh(\sqrt{2}a_0)+\frac{1}{8}\left(\cosh(\sqrt{2}a_0)-\frac{1}{\sqrt{2}a_0}\sinh(\sqrt{2}a_0)\right)(a_1+b_1)^2\Bigg]\,n^{-\frac{2}{3}}+$$

$$\cosh(2a_0^2+1)\,O(n^{-1}) \;=\; \exp(M)_{11} \;=\; \exp(M)_{22} \tag{3.46}$$

A similar argument gives

$$-\sum_{k\geq 1}\frac{2^{k-1}a^{k+1}b^{k-1}}{(2k)!} = \frac{1}{2}\left(1-\cosh(\sqrt{2}a_0)\right)+\Bigg[\frac{a_1-b_1}{2a_0}\big(1-\cosh(\sqrt{2}a_0)\big)-$$

$$\frac{a_1+b_1}{2\sqrt{2}}\sinh(\sqrt{2}a_0)\Bigg]\,n^{-\frac{1}{3}}+\Bigg[\frac{b_1(b_1-a_1)}{2a_0^2}+\frac{a_2-b_2}{2a_0}+\Bigg(\frac{b_1(a_1-b_1)}{2a_0^2}-\frac{(a_1+b_1)^2}{8}+$$

$$\frac{b_2-a_2}{2a_0}\Bigg)\cosh(\sqrt{2}a_0)+\left(\frac{5\sqrt{2}b_1^2}{16a_0}-\frac{3\sqrt{2}a_1^2}{16a_0}-\frac{\sqrt{2}a_1b_1}{8a_0}-\frac{\sqrt{2}(a_2+b_2)}{4}\right)\sinh(\sqrt{2}a_0)\Bigg]\,n^{-\frac{2}{3}}$$

$$+\cosh(\sqrt{2}a_0)\,O(n^{-1}) \;=\; \exp(M)_{12}, \tag{3.47}$$

if we interchange a and b, then

$$-\sum_{k\geq 1}\frac{2^{k-1}a^{k-1}b^{k+1}}{(2k)!} = \frac{1}{2}\left(1-\cosh(\sqrt{2}a_0)\right)+\Bigg[\frac{b_1-a_1}{2a_0}\big(1-\cosh(\sqrt{2}a_0)\big)-$$

$$\frac{a_1+b_1}{2\sqrt{2}}\sinh(\sqrt{2}a_0)\Bigg]\,n^{-\frac{1}{3}}+\Bigg[\frac{a_1(a_1-b_1)}{2a_0^2}+\frac{b_2-a_2}{2a_0}+\Bigg(\frac{a_1(b_1-a_1)}{2a_0^2}-\frac{(a_1+b_1)^2}{8}+$$

$$\frac{a_2-b_2}{2a_0}\Bigg)\cosh(\sqrt{2}a_0)+\left(\frac{5\sqrt{2}a_1^2}{16a_0}-\frac{3\sqrt{2}b_1^2}{16a_0}-\frac{\sqrt{2}a_1b_1}{8a_0}-\frac{\sqrt{2}(a_2+b_2)}{4}\right)\sinh(\sqrt{2}a_0)\Bigg]\,n^{-\frac{2}{3}}$$

$$+\cosh(\sqrt{2}a_0)\,O(n^{-1}) \;=\; \exp(M)_{21}, \tag{3.48}$$

We also have

$$\sum_{k\geq 0}\frac{2^k a^{k+1}b^k}{(2k+1)!} = \frac{1}{\sqrt{2}}\sinh(\sqrt{2}a_0)+\left[\frac{a_1+b_1}{2}\cosh(\sqrt{2}a_0)+\frac{a_1-b_1}{2\sqrt{2}a_0}\sinh(\sqrt{2}a_0)\right]n^{-\frac{1}{3}}+$$

$$\left[\left(\frac{(a_1+b_1)^2}{8a_0}-\frac{b_1^2}{2a_0}+\frac{a_2+b_2}{2}\right)\cosh(\sqrt{2}a_0)+\left(\frac{(a_1+b_1)^2}{4\sqrt{2}}-\frac{(a_1+b_1)^2}{8\sqrt{2}a_0^2}+\frac{b_1^2}{2\sqrt{2}a_0^2}\right.\right.$$
$$\left.\left.+\ \frac{a_2-b_2}{2\sqrt{2}a_0}\right)\sinh(\sqrt{2}a_0)\right]n^{-\frac{2}{3}}+\sinh(\sqrt{2}a_0)O(n^{-1}) \ = \ \exp(M)_{13} \ = \ -\exp(M)_{32}, \tag{3.49}$$

and if we interchange a and b in this last formula,

$$\sum_{k\geq 0}\frac{2^k a^k b^{k+1}}{(2k+1)!}=\frac{1}{\sqrt{2}}\sinh(\sqrt{2}a_0)+\left[\frac{a_1+b_1}{2}\cosh(\sqrt{2}a_0)+\frac{b_1-a_1}{2\sqrt{2}a_0}\sinh(\sqrt{2}a_0)\right]n^{-\frac{1}{3}}+$$
$$\left[\left(\frac{(a_1+b_1)^2}{8a_0}-\frac{a_1^2}{2a_0}+\frac{a_2+b_2}{2}\right)\cosh(\sqrt{2}a_0)+\left(\frac{(a_1+b_1)^2}{4\sqrt{2}}-\frac{(a_1+b_1)^2}{8\sqrt{2}a_0^2}+\frac{a_1^2}{2\sqrt{2}a_0^2}\right.\right.$$
$$\left.\left.+\ \frac{b_2-a_2}{2\sqrt{2}a_0}\right)\sinh(\sqrt{2}a_0)\right]n^{-\frac{2}{3}}+\sinh(\sqrt{2}a_0)O(n^{-1}) \ = \ \exp(M)_{31} \ = \ -\exp(M)_{23}. \tag{3.50}$$

The last term of the exponential matrix (3.33) is

$$1+\sum_{k\geq 1}\frac{2^k a^k b^k}{(2k)!}=\cosh(\sqrt{2}a_0)+\frac{a_1+b_1}{\sqrt{2}}\sinh(\sqrt{2}a_0)\,n^{-\frac{1}{3}}\ +\ \left[\frac{(a_1+b_1)^2}{4}\cosh(\sqrt{2}a_0)+\right.$$
$$\left.\left(\frac{a_1b_1}{\sqrt{2}a_0}+\frac{a_2+b_2}{\sqrt{2}}-\frac{(a_1+b_1)^2}{4\sqrt{2}a_0}\right)\sinh(\sqrt{2}a_0)\right]n^{-\frac{2}{3}}+\cosh(\sqrt{2}a_0)\,O(n^{-1}) \ = \ \exp(M)_{33}. \tag{3.51}$$

We note that $\sqrt{2}a_0$ is exactly the quantity μ defined in [24] as the new variable when solving for the limiting system of equation as n goes to infinity.

We then use equations (3.35), (3.46), (3.50) together with the numerical value of c_φ for n even given by (2.22) to have the following expansion of $V_{n,\varepsilon}$ and $q_{n,\varepsilon}$.

$$V_{n,\varepsilon}(\tau(s)) = \frac{1}{2}(1-\exp(-\sqrt{2}a_0))+\left[\frac{a_1-b_1}{2\sqrt{2}}\frac{\sinh(\sqrt{2}a_0)}{\sqrt{2}a_0}-\frac{a_1+b_1}{2\sqrt{2}}\exp(-\sqrt{2}a_0)\right]n^{-\frac{1}{3}}+$$
$$\left(\left[\frac{(a_1+b_1)^2}{8}(1-\frac{1}{\sqrt{2}a_0})-\frac{\sqrt{2}(a_2+b_2)}{4}+\frac{a_1^2}{2\sqrt{2}a_0}\right]\cosh(\sqrt{2}a_0)-\left[\frac{(a_1+b_1)^2}{8}(1+\frac{1}{\sqrt{2}a_0})\right.\right.$$
$$\left.\left.-\frac{(a_1+b_1)^2}{16a_0^2}-\frac{a_1b_1}{2\sqrt{2}a_0}+\frac{a_0^2}{4a_0^2}-\frac{\sqrt{2}(a_2-b_2)}{4\sqrt{2}a_0}-\frac{\sqrt{2}(a_2+b_2)}{4}\right]\sinh(\sqrt{2}a_0)\right)n^{-\frac{2}{3}}+O(\frac{1}{n})$$

and

$$q_{n,\varepsilon}(\tau(s)) \ = \ \frac{1}{\sqrt{2}}\exp(-\sqrt{2}a_0)+\left[-\frac{a_1+b_1}{2}\exp(-\sqrt{2}a_0)-\frac{a_1-b_1}{2}\frac{\sinh(\sqrt{2}a_0)}{\sqrt{2}a_0}\right]n^{-\frac{1}{3}}$$
$$\left(\left[\frac{\sqrt{2}(a_1+b_1)^2}{8}(1-\frac{1}{\sqrt{2}a_0})+\frac{b_1^2}{2a_0}-\frac{a_2+b_2}{2}\right]\cosh(\sqrt{2}a_0)+\left[\frac{b_2-a_2}{2\sqrt{2}a_0}+\frac{a_2+b_2}{2}\right.\right.$$
$$\left.\left.-\frac{\sqrt{2}(a_1+b_1)^2}{8}(1+\frac{1}{\sqrt{2}a_0})+\frac{a_1b_1}{2a_0}+\frac{\sqrt{2}(a_1+b_1)^2}{16a_0^2}-\frac{\sqrt{2}b_1^2}{4a_0^2}\right]\sinh(\sqrt{2}a_0)\right)n^{-\frac{2}{3}}+O(\frac{1}{n}).$$

3.2.2 Second system of equations involving the calligraphic variables for $\mathbf{GOE}_n$

The system involving the calligraphic variables is

$$Y(t) = \exp\left(-\int_t^\infty B(x)dx\right) \cdot Y(\infty). \tag{3.52}$$

We set

$$\exp\left(-\int_t^\infty B(x)\,dx\right) = \exp\left\{\begin{pmatrix} 0 & 0 & -\int_t^\infty q_n(x)\,dx \\ 0 & 0 & -\int_t^\infty p_n(x)\,dx \\ -\int_t^\infty p_n(x)dx & -\int_t^\infty q_n(x)\,dx & 0 \end{pmatrix}\right\}$$
$$= \exp(\mathcal{M}).$$

$\mathcal{M}$ is of the form

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & -b \\ -b & -a & 0 \end{pmatrix} \tag{3.53}$$

with

$$\exp(\mathcal{M}) = \begin{pmatrix} 1+\sum_{k\geq 1}\frac{2^{k-1}a^kb^k}{(2k)!} & \sum_{k\geq 1}\frac{2^{k-1}a^{k+1}b^{k-1}}{(2k)!} & -\sum_{k\geq 0}\frac{2^ka^{k+1}b^k}{(2k+1)!} \\ \sum_{k\geq 1}\frac{2^{k-1}a^{k-1}b^{k+1}}{(2k)!} & 1+\sum_{k\geq 1}\frac{2^{k-1}a^kb^k}{(2k)!} & -\sum_{k\geq 0}\frac{2^ka^kb^{k+1}}{(2k+1)!} \\ -\sum_{k\geq 0}\frac{2^ka^kb^{k+1}}{(2k+1)!} & -\sum_{k\geq 0}\frac{2^ka^{k+1}b^k}{(2k+1)!} & 1+\sum_{k\geq 1}\frac{2^ka^kb^k}{(2k)!} \end{pmatrix}$$

$$= \begin{pmatrix} \exp(\mathcal{M})_{11} & \exp(\mathcal{M})_{12} & \exp(\mathcal{M})_{13} \\ \exp(\mathcal{M})_{21} & \exp(\mathcal{M})_{22} & \exp(\mathcal{M})_{23} \\ \exp(\mathcal{M})_{31} & \exp(\mathcal{M})_{32} & \exp(\mathcal{M})_{33} \end{pmatrix} \tag{3.54}$$

We have

$$\mathcal{Q}_{n,1}(t) = 2c_\varphi \exp(\mathcal{M})_{11} + \exp(\mathcal{M})_{13}, \tag{3.55}$$
$$\mathcal{P}_{n,1}(t) = 2c_\varphi \exp(\mathcal{M})_{21} + \exp(\mathcal{M})_{23}, \tag{3.56}$$

and

$$\tilde{\mathcal{R}}_{n,1}(t) = 2c_\varphi \exp(\mathcal{M})_{31} + \exp(\mathcal{M})_{33}. \tag{3.57}$$

We note that $\exp(\mathcal{M}_{21}) = -\exp(M)_{21}$, $\exp(\mathcal{M}_{23}) = \exp(M)_{23}$, $\exp(\mathcal{M}_{31}) = -\exp(M)_{31}$ and $\exp(\mathcal{M}_{33}) = \exp(M)_{33}$. The solutions (3.56) and (3.57) follow directly from the large n expansion obtained in the last subsection. We therefore have the following solutions for $\mathcal{P}_{n,1}$ and $\tilde{\mathcal{R}}_{n,1}$

$$\mathcal{P}_{n,1}(\tau(s)) = \frac{1}{\sqrt{2}}(\exp(-\sqrt{2}a_0)-1)+\Bigg[\frac{a_1-b_1}{\sqrt{2}a_0}+\frac{a_1+b_1}{2}\exp(-\sqrt{2}a_0)+\frac{b_1-a_1}{\sqrt{2}a_0}\cosh(\sqrt{2}a_0)$$

$$- \frac{b_1 - a_1}{\sqrt{2}a_0}\frac{\sinh(\sqrt{2}a_0)}{2})\Bigg] n^{-\frac{1}{3}} + \Bigg(-\frac{\sqrt{2}a_1^2}{2a_0^2} + \frac{\sqrt{2}a_1 b_1}{2a_0^2} + \frac{a_2 - b_2}{\sqrt{2}a_0} + \Bigg[\frac{(a_1+b_1)^2}{4\sqrt{2}} + \frac{\sqrt{2}(a_1^2 + a_1 b_1)}{2a_0^2} .$$

$$- \frac{a_1^2}{4a_0} - \frac{a_2 - b_2}{\sqrt{2}a_0} + \frac{b_1^2}{4a_0}\Bigg] \cosh(\sqrt{2}a_0) + \Bigg[-\frac{(a_1+b_1)^2}{4\sqrt{2}} + \frac{(a_1+b_1)^2}{8\sqrt{2}a_0^2} - \frac{a_1^2}{2\sqrt{2}a_0^2}$$

$$+ \frac{a_2 - b_2}{2\sqrt{2}a_0}\Bigg] \sinh(\sqrt{2}a_0)\Bigg) n^{-\frac{2}{3}} + O(\frac{1}{n}) \tag{3.58}$$

and

$$\tilde{\mathcal{R}}_{n,1}(\tau(s)) = \exp(-\sqrt{2}a_0) + \Bigg[\frac{a_1 - b_1}{\sqrt{2}} \frac{\sinh(\sqrt{2}a_0)}{\sqrt{2}a_0} - \frac{a_1 + b_1}{\sqrt{2}} \exp(-\sqrt{2}a_0)\Bigg] n^{-\frac{1}{3}}$$

$$\Bigg[\Bigg(\frac{(a_1+b_1)^2}{4} - \frac{a_2 + b_2}{\sqrt{2}} \Bigg) \exp(-\sqrt{2}a_0) - \frac{(a_1+b_1)^2}{4\sqrt{2}a_0} \cosh(\sqrt{2}a_0) - \frac{(a_1-b_1)^2}{4\sqrt{2}a_0} \sinh(\sqrt{2}a_0)$$

$$\frac{(a_1+b_1)^2}{8a_0^2} \sinh(\sqrt{2}a_0) + \frac{a_2 - b_2}{2a_0} \sinh(\sqrt{2}a_0) + \frac{a_1^2}{\sqrt{2}a_0} \cosh(\sqrt{2}a_0) - \frac{a_1^2}{2a_0^2} \sinh(\sqrt{2}a_0)\Bigg] n^{-\frac{2}{3}}$$

$$+ O(\frac{1}{n}) \tag{3.59}$$

3.2.3 Calligraphic variables for GSE$_n$

We note that the GSE$_n$ case is identical to the GOE$_n$ up to a sign change and the parity of n for the calligraphic variables. The large n expansion for $u_{n,\varepsilon}(t)$ and $\tilde{v}_{n,\varepsilon}(t)$ follows from the matrix exponential (3.33). The boundary conditions need to be change to $u_{n,\varepsilon}(\infty) = 0$ and $\tilde{v}_{n,\varepsilon}(\infty) = 0$ and $q_{n,\varepsilon}(\infty) = 0$ as n is odd. Therefore in this case

$$u_{n,\varepsilon}(t) = \exp(M)_{12} \tag{3.60}$$

$$V_{n,\varepsilon}(t) = \exp(M)_{22} \tag{3.61}$$

and

$$q_{n,\varepsilon}(t) = \exp(M)_{32} \tag{3.62}$$

The large n expansions of these quantities is given by (3.47), (3.46) and (3.49) respectively.

The system of equations satisfied by the calligraphic variables is

$$\frac{d}{dt} \begin{pmatrix} \mathcal{Q}_{n,4}(t) \\ \mathcal{P}_{n,4}(t) \\ \tilde{\mathcal{R}}_{n,4}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -q_n(t) \\ 0 & 0 & -p_n(t) \\ -p_n(t) & -q_n(t) & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathcal{Q}_{n,4}(t) \\ \mathcal{P}_{n,4}(t) \\ \tilde{\mathcal{R}}_{n,4}(t) \end{pmatrix}. \tag{3.63}$$

where $\tilde{\mathcal{R}}_{n,4}(t) = 1 + \mathcal{R}_{n,4}(t)$. The boundary conditions in this case are

$$\begin{pmatrix} \mathcal{Q}_{n,4}(\infty) \\ \mathcal{P}_{n,4}(\infty) \\ \tilde{\mathcal{R}}_{n,4}(\infty) \end{pmatrix} = \begin{pmatrix} -c_\varphi \\ -c_\psi \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -c_\psi \\ 1 \end{pmatrix} \quad \text{as } n \text{ is odd} \tag{3.64}$$

We can use the same technique as for the Orthogonal case to find a large n expansion of $\mathcal{Q}_{n,4}(t)$, $\mathcal{P}_{n,4}(t)$ and $\tilde{\mathcal{R}}_{n,4}(t)$.

3.3 Large n expansion of the probability distribution of the largest eigenvalue

We recall that the quantity of interest is (2.24). Under our change of variables it reads

$$(1-\tilde{v}_{n,\varepsilon})(1-\frac{1}{2}\mathcal{R}_{n,1})-\frac{1}{2}(q_{n,\varepsilon}-c_\varphi)\mathcal{P}_{n,1} = \frac{1}{2}\left[V_{n,\varepsilon}\big(1+\tilde{\mathcal{R}}_{n,1}\big)-\mathcal{P}_{n,1}\big(q_{n,\varepsilon}-c_\varphi\big)\right]. \quad (3.65)$$

Upon substitutions of the newly derived expressions in the right of (3.65), the right side of (3.65) takes the form

$$\exp(-\sqrt{2}a_0)+\left[\frac{a_1-b_1}{2a_0}(1-\exp(-\sqrt{2}a_0))-\frac{a_1+b_1}{\sqrt{2}}\exp(-\sqrt{2}a_0)\right]n^{-\frac{1}{3}}$$

$$\Bigg\{\frac{a_1b_1-a_1^2}{2a_0^2}+\frac{a_2-b_2}{2a_0}+\left(\frac{3a_1^2-b_1^2}{8\sqrt{2}a_0}-\frac{(a_1+b_1)^2}{16\sqrt{2}a_0}-\frac{a_2+b_2}{4\sqrt{2}}\right)\exp(-2\sqrt{2}a_0)$$

$$+\left(\frac{(a_1+b_1)^2}{4}-\frac{3(a_2+b_2)}{4\sqrt{2}}+\frac{a_1^2}{2a_0^2}-\frac{(a_1+b_1)^2}{16\sqrt{2}a_0}+\frac{a_1^2}{2\sqrt{2}a_0}-\frac{a_1b_1}{4\sqrt{2}a_0}-\frac{a_2-b_2}{2a_0}\right)\exp(-\sqrt{2}a_0)$$

$$\frac{(a_1+b_1)^2}{8a_0^2}\sinh(\sqrt{2}a_0)-\frac{a_1b_1}{2a_0^2}\cosh(\sqrt{2}a_0)\Bigg\}n^{-\frac{2}{3}}+O(\frac{1}{n}) \quad (3.66)$$

We follow Tracy and Widom [24] and denote by $\mu(s)$ the quantity $\sqrt{2}a_0(s)$,

$$\mu := \mu(s) \;=\; \sqrt{2}a_0(s) = \int_s^\infty q(x)dx, \quad (3.67)$$

and we introduce the following notations,

$$\nu := \nu(s) \;=\; \int_s^\infty p(x)dx, \quad \alpha := \alpha(s) \;=\; \int_s^\infty q(x)\,u(x)\,dx, \quad (3.68)$$

$$a_1(s) = \frac{1}{\sqrt{2}}\int_s^\infty\left(\frac{2c-1}{2}p(x)-cq(x)u(x)\right)dx,$$

$$b_1(s) = \frac{1}{\sqrt{2}}\int_s^\infty\left(\frac{2c+1}{2}p(x)-cq(x)u(x)\right)dx,$$

$$a_2(s) = \frac{1}{20\sqrt{2}}\int_s^\infty\Bigg(\big(10c^2-10c+\frac{3}{2}\big)q_1(x)+p_2(x)+\big(-30c^2+10c+\frac{3}{2}\big)q(x)v(x)+p_1v(x)$$

$$+p(x)v_1(x)-q_2(x)u(x)-q_1(x)u_1(x)-q(x)u_2(x)+\big(\frac{3}{2}-10c^2\big)p(x)u(x)+20c^2q(x)u^2(x)\Bigg)\,dx,$$

$$b_2(s) = \frac{1}{20\sqrt{2}}\int_s^\infty\Bigg(\big(10c^2+10c+\frac{3}{2}\big)q_1(x)+p_2(x)+\big(-30c^2-10c+\frac{3}{2}\big)q(x)v(x)+p_1v(x)$$

$$+p(x)v_1(x) - q_2(x)u(x) - q_1(x)u_1(x) - q(x)u_2(x) + \big(\frac{3}{2} - 10c^2\big)p(x)u(x) + 20c^2 q(x)u^2(x)\Big)\ dx.$$

We note that

$$a_1(s) - b_1(s) \ = \ -\frac{1}{\sqrt{2}}\int_s^\infty p(x)\,dx, \tag{3.69}$$

$$a_1(s) + b_1(s) \ = \ \frac{1}{\sqrt{2}}\int_s^\infty \big(2cp(x) - 2cq(x)u(x)\big)dx \ = \ -\sqrt{2}\,c\,q(s), \tag{3.70}$$

and

$$a_2(s) - b_2(s) = \frac{c}{\sqrt{2}}\int_s^\infty \big(q(x)v(x) - q_1(x)\big)dx \ = \ \frac{c}{\sqrt{2}}p(s). \tag{3.71}$$

We set

$$\eta(s) = a_2(s)+b_2(s) = \frac{1}{20\sqrt{2}}\int_s^\infty \Big(\big(20c^2{+}3\big)q_1(x){+}2p_2(x){+}\big({-}60c^2{+}3\big)q(x)v(x){+}2p_1v(x)$$

$$+2p(x)v_1(x) - 2q_2(x)u(x) - 2q_1(x)u_1(x) - 2q(x)u_2(x) + \big(3 - 20c^2\big)p(x)u(x)$$

$$+ 40c^2 q(x)u^2(x)\Big)\,dx. \tag{3.72}$$

$$= -\frac{20c^2q'(s) + 3p(s)}{20\sqrt{2}} + \frac{1}{20\sqrt{2}}\int_s^\infty \big(6qv{+}3pu{+}2p_2{+}2p_1v{+}2pv_1{-}2q_2u{-}2q_1u_1{-}2qu_2\big)(x)\,dx. \tag{3.73}$$

This last equality comes from

$$(20c^2+3)q_1(x) + (3-60c^2)q(x)v(x) + (3-20c^2)p(x)u(x) + 40c^2q(x)u^2(x) \ =$$

$$20c^2\big(q_1(x) - 3q(x)v(x) - p(x)u(x) + 2q(x)u^2(x)\big) + 3\big(q_1(x) + q(x)v(x) + p(x)u(x)\big), \tag{3.74}$$

the substitutions

$$q_1(x) = xq(x) - q(x)v(x) + p(x)u(x), \quad \text{and} \quad q_1(x) = p'(x) + q(x)v(x), \tag{3.75}$$

in (3.74)

$$20c^2\big(xq(x) - 4q(x)v(x) + 2q(x)u^2(x)\big) + 3\big(p'(x) + 2q(x)v(x) + p(x)u(x)\big)$$

$$= \ 20c^2\big(xq(x) + 2q(x)\big(-2v(x) + u^2(x)\big)\big) + 3\big(p'(x) + 2q(x)v(x) + p(x)u(x)\big), \tag{3.76}$$

and the substitutions

$$q^2(x) = u^2(x) - 2v(x), \quad \text{and} \quad q''(x) = xq(x) + 2q^3(x) \tag{3.77}$$

in (3.76)

$$20c^2\big(xq(x) + 2q^3(x)\big) + 3\big(p'(x) + 2q(x)v(x) + p(x)u(x)\big)$$

$$= \ 20c^2q''(x) + 3p'(x) + 6q(x)v(x) + 3p(x)u(x). \tag{3.78}$$

With these representations, (3.66) is

$$e^{-\mu}+\left[c\,q(s)e^{-\mu}\;-\;\frac{1}{2\mu}\nu\big(1-e^{-\mu}\big)\right]n^{-\frac{1}{3}}+\left\{\frac{2c-1}{4\mu^2}\nu^2-\frac{c}{2\mu^2}\nu\int_s^\infty q(x)u(x)dx+\frac{c\,p(s)}{2\mu}\right.$$

$$\left[\frac{c^2-2c+1/4}{8\mu}\nu^2-\frac{c(c-1)}{4\mu}\nu\int_s^\infty q(x)u(x)dx+\frac{c^2}{8\mu}\left(\int_s^\infty q(x)u(x)dx\right)^2-\frac{c^2\,q^2(s)}{8\mu}\right.$$

$$\left.-\frac{\eta}{4\sqrt{2}}\right]e^{-2\mu}+\left[\frac{c^2\,q^2(s)}{2}-\frac{3\eta}{4\sqrt{2}}+\frac{2-\mu}{2\mu^2}\left(\frac{c^2-c+1/4}{2}\nu^2-\frac{2c^2-c}{2}\nu\int_s^\infty q(x)u(x)dx\right.\right.$$

$$\left.+\frac{c^2}{2}\Big(\int_s^\infty q(x)u(x)dx\Big)^2\right)-\frac{c^2\,q^2(s)}{8\mu}-\frac{c^2-1/4}{8\mu}\nu^2+\frac{c^2}{4\mu}\nu\int_s^\infty q(x)u(x)dx-\frac{c\,p(s)}{2\mu}$$

$$\left.-\frac{c^2}{4\mu}\left(\int_s^\infty q(x)u(x)dx\right)^2\right]e^{-\mu}+\frac{c^2\,q^2(s)}{2\mu^2}\sinh(\mu)-\left(\frac{c^2-1/4}{2}\nu^2-c^2\nu\int_s^\infty q(x)u(x)dx\right.$$

$$\left.\left.c^2\Big(\int_s^\infty q(x)u(x)dx\Big)^2\right)\frac{\cosh(\mu)}{\mu^2}\right\}n^{-\frac{2}{3}}\;+\;O(\frac{1}{n}) \tag{3.79}$$

as n goes to infinity uniformly for s bounded away from minus infinity.

Finally we combine (3.79) with Theorem1.1 to have the following version of Theorem1.2.

If we set $t=\tau(s)$, then as $n\to\infty$

$$F_{n,1}^2(t)\;=\;F_2(s)\cdot\left\{e^{-\mu}\;+\;\left[c\big(q(s)+u(s)\big)e^{-\mu}-\frac{\nu}{2\mu}(1-e^{-\mu})\right]n^{-\frac{1}{3}}\;+\right.$$

$$\left[-\frac{1}{20}E_{c,2}(s)\,e^{-\mu}-\frac{c\,\alpha(s)}{2\mu^2}+\frac{c\,p(s)}{2\mu}+\frac{(2c-1)\,\nu^2}{4\mu^2}+c\,u(s)\left(c\,q(s)\,e^{-\mu}-\frac{\nu}{2\mu}(1-e^{-\mu})\right)\right.$$

$$+\;e^{-2\mu}\left(-\frac{\eta}{4\sqrt{2}}+\frac{c^2\,\alpha^2(s)}{8\mu}-\frac{c^2\,q^2(s)}{8\mu}-\frac{(c^2-c)\,\nu\,\alpha(s)}{4\mu}+\frac{\left(\frac{1}{4}-2c+c^2\right)\,\nu^2}{8\mu}\right)\;+$$

$$e^{-\mu}\left(\frac{c^2\,q^2(s)}{2}-\frac{3\,\eta}{4\sqrt{2}}-\frac{c^2\,\alpha^2(s)}{4\mu}-\frac{c\,p(s)}{2\mu}-\frac{c^2\,q^2(s)}{8\mu}+\frac{c^2\,\nu\,\alpha(s)}{4\mu}-\frac{\left(-\frac{1}{4}+c^2\right)\nu^2}{8\mu}+\right.$$

$$\left.\frac{2-\mu}{2\mu^2}\left(\frac{c^2\,\alpha^2(s)}{2}-\frac{(2c^2-c)\,\nu\,\alpha(s)}{2}+\frac{\left(\frac{1}{4}-c+c^2\right)\,\nu^2}{2}\right)\right)\;-$$

$$\left.\left.\left(c^2\,\alpha^2(s)-c^2\,\nu\,\alpha(s)+\frac{\left(-\frac{1}{4}+c^2\right)\,\nu^2}{2}\right)\frac{\cosh(\mu)}{\mu^2}+\frac{c^2\,q^2(s)}{8\mu^2}\sinh(\mu)\right]n^{-\frac{2}{3}}\right\}+O(n^{-1}) \tag{3.80}$$

uniformly in s.

To simplify the $n^{-\frac{2}{3}}$ term in (3.80), we use the representation $p(s)=q'(s)+q(s)u(s)$ which says in this setting that $\nu(s)=\alpha(s)-q(s)$. The result of this substitution is Theorem1.2.

4 Conclusion

We note that unlike $F_{n,2}(t)$ for the GUE_n, the GOE_n large n expansion of the probability distribution of the largest eigenvalue $F_{n,1}(t)$ has a non vanishing $n^{-\frac{1}{3}}$ correction term. Thus the convergence to the limiting Tracy-Widom distribution $F_1(t)$ is slower. Numerical applications of $F_{n,1}(t)$ follows easily from $q(s)$ this is one consequence of our representation of $F_{n,1}(t)$ in Theorem1.2. All the terms on the right side of (1.36) can be expressed in terms of $q(s)$ and $q'(s)$.
The GSE_n largest eigenvalue distribution is derived in a similar way (the only major difference being that n needs to be odd in this case.)

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