Edgeworth Expansion of the Largest Eigenvalue Distribution Function of GUE and LUE

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Abstract

We derive expansions of the Hermite and Laguerre kernels at the edge of the spectrum of the finite n Gaussian Unitary Ensemble (GUE_n) and the finite n Laguerre Unitary Ensemble (LUE_n), respectively. Using these large n kernel expansions, we prove an Edgeworth type theorem for the largest eigenvalue distribution function of GUE_n and LUE_n. In our Edgeworth expansion, the correction terms are expressed in terms of the same Painlevé II function appearing in the leading term, i.e. in the Tracy-Widom distribution. We conclude with a brief discussion of the universality of these results.

1 Introduction

The limiting distribution function for the largest eigenvalues in orthogonal, unitary and symplectic random matrix ensembles have found many applications outside their initial discovery in random matrix theory, see, for example, [3, 4, 24] for recent reviews. In these applications it is important to have correction terms to the limiting distribution. For example, in statistics [12] the sample size is always finite; and to assess quantitatively the range of validity of limit laws, one needs finite n correction terms. In classical probability, a similar issue arises in the application of the Central Limit Theorem (CLT) to finite n problems. Here the two main results are the Berry-Esseen theorem and the Edgeworth expansion [6].

Recall if S_n is a sum of i.i.d. random variables X_j, each with mean μ and variance σ^2, that the distribution F_n of the normalized random variable $(S_n - n\mu)/(\sigma\sqrt{n})$

satisfies the Edgeworth expansion[1]

$$F_n(x) - \Phi(x) = \phi(x) \sum_{j=3}^{r} n^{-\frac{1}{2}j+1} R_j(x) + o(n^{-\frac{1}{2}r+1}) \tag{1.1}$$

uniformly in x. Here Φ is the standard normal distribution with density ϕ, and R_j are polynomials depending only on $\mathbb{E}(X_j^k)$ but not on n and r (or the underlying distribution of the X_j).

Introduce

$$F_{n,2}^{G,L}(t) = \mathbb{P}_{G,L}(\lambda_{\max}^{G,L} \leq t) \tag{1.2}$$

where $\lambda_{\max}^{G,L}$ is the largest eigenvalue in GUE_n or LUE_n, respectively. (When the context is clear, we often drop the G or L.) To obtain a nontrivial limit theorem, we must, as is well known, define normalized random variables $\hat{\lambda}_{\max}^{G,L}$. We find it useful to "fine tune" our normalization (see also [12]),

$$\begin{aligned} \hat{\lambda}_{\max}^{G} &:= \frac{\lambda_{\max}^{G} - (2(n+c_G))^{1/2}}{2^{-1/2} n^{-1/6}}, && (1.3) \\ \hat{\lambda}_{\max}^{L} &:= \frac{\lambda_{\max}^{L} - 4(n+c_L) - 2\alpha}{2(2n)^{1/3}} && (1.4) \end{aligned}$$

where $c_{G,L}$ are constants (to be chosen later), and α is the parameter appearing in LUE_n. (That is, the parameter α appearing in the Laguerre polynomials L_n^{α}.) Then $\hat{\lambda}_{\max}^{G,L}$ converge in distribution to GUE Tracy-Widom (commonly denoted F_2). In this paper we initiate the study of Edgeworth expansions for both GUE_n and LUE_n; that is, we find the analogue of (1.1) for $F_n^{G,L} - F_2$. We now state our main results.

Our first result is an extension of the Plancherel-Rotach theorem for the Laguerre poynomials L_n^{α}. We set

$$\xi = (4n + 2\alpha + 2c)^{\frac{1}{2}} + \frac{X}{2^{\frac{2}{3}} n^{\frac{1}{6}}} \quad \text{where } X \text{ and } c \text{ are bounded,} \tag{1.5}$$

and denote by Ai the Airy function (see, e.g., [15]).

Theorem 1.1. *For $\alpha > -1$ we have as $n \to \infty$*

$$\begin{aligned} e^{-\xi^2/2} L_n^{\alpha}(\xi^2) = (-1)^n 2^{-\alpha-\frac{1}{3}} n^{-\frac{1}{3}} \quad &\Bigg\{ \mathrm{Ai}(X) + \frac{(c-1)}{2^{\frac{1}{3}}} \mathrm{Ai}'(X) n^{-\frac{1}{3}} + \\ &\left[\frac{2 - 10c + 5c^2 - 5\alpha}{10 \cdot 2^{\frac{2}{3}}} X \,\mathrm{Ai}(X) + \frac{X^2}{20 \cdot 2^{\frac{2}{3}}} \mathrm{Ai}'(X) \right] n^{-\frac{2}{3}} + \\ &\left[\left(\frac{5\alpha - 15c\alpha + 2c^3 - 15c^2 - 56c - 6}{60} + \frac{c-1}{40} X^3 \right) \mathrm{Ai}(X) + \right. \\ &\left. \frac{(c-1)(5(c-2)c - 3(2+5\alpha))}{60} X \,\mathrm{Ai}'(X) \right] n^{-1} + O(n^{-\frac{4}{3}}) \,\mathrm{Ai}(X) \Bigg\} \end{aligned}$$

[1]We assume, of course, the moments $\mathbb{E}(X_j^k)$, $k = 3, \ldots, r$, exist; and as well, the condition $\lim_{|\zeta| \to \infty} \sup |\varphi(\zeta)| < \infty$ where φ is the characteristic function of X_j, see [6].

From Theorem 1.1 we derive expansions for both the Hermite and Laguerre kernels. Recall that if[2]

$$\varphi_n(x) = \frac{1}{(2^n n! \sqrt{\pi})^{1/2}} H_n(x)\, e^{-x^2/2} \quad \text{and} \quad \phi_n^\alpha(x) = x^{\alpha/2}\, e^{-x/2}\, L_n^\alpha(x),$$

then the Hermite kernel is

$$K_n(x,y) = \sum_{k=0}^{n-1} \varphi_k(x)\varphi_k(y) \tag{1.6}$$

and the Laguerre kernel is

$$K_n^\alpha(x,y) = \sum_{k=0}^{n-1} \frac{\phi_k^\alpha(x)\phi_k^\alpha(y)}{\Gamma(k+1)\Gamma(\alpha+k+1)}\,. \tag{1.7}$$

Finally, the Airy kernel is

$$K_{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}(y)\,\mathrm{Ai}'(x)}{x-y} = \int_0^\infty \mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz. \tag{1.8}$$

Using Theorem 1.1 we prove

Theorem 1.2. *For $x = (2(n+c_G))^{\frac{1}{2}} + 2^{-\frac{1}{2}} n^{-\frac{1}{6}} X$ and $y = (2(n+c_G))^{\frac{1}{2}} + 2^{-\frac{1}{2}} n^{-\frac{1}{6}} Y$ with X, Y and c_G bounded,*

$$K_n(x,y)\,dx = \Big\{ K_{\mathrm{Ai}}(X,Y) - c_G\,\mathrm{Ai}(X)\,\mathrm{Ai}(Y) n^{-\frac{1}{3}} +$$

$$\frac{1}{20}\Big[(X+Y)\,\mathrm{Ai}'(X)\,\mathrm{Ai}'(Y) - (X^2+XY+Y^2)\,\mathrm{Ai}(X)\,\mathrm{Ai}(Y) +$$

$$\frac{-20c_G^2+3}{2}(\mathrm{Ai}'(X)\,\mathrm{Ai}(Y) + \mathrm{Ai}(X)\,\mathrm{Ai}'(Y))\Big]\, n^{-\frac{2}{3}} + O(n^{-1})E(X,Y)\Big\}\, dX. \tag{1.9}$$

The error term, $E(X,Y)$, is the kernel of an integral operator on $L^2(J)$ which is trace class for any Borel subset J of the reals that is bounded away from minus infinity.

For the Laguerre kernel we prove

Theorem 1.3. *For $x = 4(n+c_L)+2\alpha+2(2n)^{\frac{1}{3}}X$ and $y = 4(n+c_L)+2\alpha+2(2n)^{\frac{1}{3}}Y$ with X, Y and c_L bounded,*

$$K_n^\alpha(x,y)\,dx = \Big\{ K_{\mathrm{Ai}}(X,Y) - 2^{\frac{2}{3}} c_L\,\mathrm{Ai}(X)\,\mathrm{Ai}(Y) n^{-\frac{1}{3}} +$$

[2]Here $H_n(x)$ are the Hermite polynomials of degree n.

$$\frac{2^{\frac{1}{3}}}{10}\Big[\quad (X^2+XY+Y^2)\operatorname{Ai}(X)\operatorname{Ai}(Y)-(X+Y)\operatorname{Ai}'(X)\operatorname{Ai}'(Y)-$$

$$(10c_L^2-1)(\operatorname{Ai}(X)\operatorname{Ai}'(Y)+\operatorname{Ai}'(X)\operatorname{Ai}(Y))\Big]\,n^{-\frac{2}{3}}+O(n^{-1})F(X,Y)\Big\}\,dX. \tag{1.10}$$

The error term $F(X,Y)$ is the kernel of an integral operator on $L^2(J)$ which is trace class for any Borel subset J of the reals which is bounded away from minus infinity.

To state our main theorem, we need a number of definitions. First define the constants

$$a^G_{c_G,2}=c_G,\quad a^L_{c_L,2}=2^{\frac{2}{3}}c_L,\quad b^G_2=-\frac{1}{20},\quad b^L_2=\frac{2^{\frac{1}{3}}}{10}.$$

Following the notations of Tracy and Widom [20], we set

$$E^G_{c_G,2}(s)=2w_1-3u_2+(-20c_G^2+3)v_0+u_1v_0-u_0v_1+u_0v_0^2-u_0^2w_0,$$

$$E^L_{c_L,2}(s)=2w_1-3u_2+(20c_L^2-2)v_0+u_1v_0-u_0v_1+u_0v_0^2-u_0^2w_0,$$

where

$$u_i:=u_i(s)=\int_s^\infty q(x)x^i\operatorname{Ai}(x)\,dx,\quad v_i:=v_i(s)=\int_s^\infty q(x)x^i\operatorname{Ai}'(x)\,dx\quad\text{and}$$

$$w_i:=w_i(s)=\int_s^\infty q'(x)x^i\operatorname{Ai}'(x)\,dx\;+\;u_0(s)v_i(s)$$

with q the solution of the Painlevé II equation, $q''=sq+2q^3$, subject to the boundary condition $q(s)\sim\operatorname{Ai}(s)$ as $s\to\infty$. Finally, the GUE Tracy-Widom distribution is

$$F_2(s)=\det\big(I-K_{\mathrm{Ai}}\chi_{(s,\infty)}\big)=\exp\left(-\int_s^\infty (x-s)q(x)^2\,dx\right) \tag{1.11}$$

where K_{Ai} is the operator with Airy kernel (1.8) and $\chi_{(s,\infty)}$ is the indicator function of the interval (s,∞). With now can state our main result.

Theorem 1.4. *We set*

$$t=(2(n+c_G))^{\frac{1}{2}}+2^{-\frac{1}{2}}n^{-\frac{1}{6}}\,s\quad\textit{for } GUE_n \tag{1.12}$$

and

$$t=4(n+c_L)+2\alpha+2(2n)^{\frac{1}{3}}\,s\quad\textit{for } LUE_n\,. \tag{1.13}$$

Then as $n\to\infty$

$$F^{G,L}_{n,2}(t)=F_2(s)\{1+a^{G,L}_{c_{G,L},2}\,u_0(s)\,n^{-\frac{1}{3}}+b^{G,L}_2E^{G,L}_{c_{G,L},2}(s)\,n^{-\frac{2}{3}}\}+O(n^{-1}) \tag{1.14}$$

uniformly in s. If in addition,

$$c_G^2+c_L^2=\frac{1}{4},\ \textit{ then } E^G_{c_G,2}(s)=E^L_{c_L,2}(s)=E_{c,2}(s),\ \textit{ and}$$

$$F^{G,L}_{n,2}(t)=F_2(s)\{1+a^{G,L}_{c_{G,L},2}\,u_0(s)\,n^{-\frac{1}{3}}+b^{G,L}_2E_{c,2}(s)\,n^{-\frac{2}{3}}\}+O(n^{-1}). \tag{1.15}$$

Note that the $n^{-\frac{1}{2}}$ correction term in the Edgeworth expansion for the CLT is universal in the sense that only the constant factor depends on the underlying distribution. We see in (1.15) a similar universality, and we conjecture that this universality extends to a wider class of unitary ensembles.

In §2 we derive Theorems 1.2 and 1.3 from Theorem 1.1. In §3 we follow [20] to prove Theorem 1.4. The proof of Theorem 1.1 will be given in the Appendices together with some facts needed to prove the last Theorem.

2 Correction terms for the Hermite and Laguerre kernel at the edge of the spectrum

To simplify notations we will use matrix ensembles of $(n+1)\times(n+1)$ matrices throughout this section and part of the next section. After the fine tuning of the variables in §3.3, we will use ensembles of $n \times n$ matrices.

2.1 Hermite case

We have the following representation of the Hermite kernel from the Christoffel-Darboux formula.

$$K_{n+1}(x,y) = \sqrt{\frac{n+1}{2}} \quad \frac{\varphi_{n+1}(x)\varphi_n(y) - \varphi_{n+1}(y)\varphi_n(x)}{x-y} \tag{2.1}$$

As mentioned in §1, Theorem 1.2 is a corollary of Theorem 1.1. We recall the relation between the Hermite and Laguerre polynomials

$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{-\frac{1}{2}}(x^2), \;\; H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{\frac{1}{2}}(x^2)$$

so as to estimate the right side of (2.1). We will also assume[3] without lost of generalities that $n = 2k$ is even. Using the symmetry of the Hermite kernel in (2.1) we only need to find an expansion of

$$\varphi_{n+1}(x)\varphi_n(y) = \frac{2^{4k+1}(k!)^2 x e^{-\frac{x^2}{2}} L_k^{\frac{1}{2}}(x^2) e^{-\frac{y^2}{2}} L_k^{-\frac{1}{2}}(y^2)}{(2^{4k+1}(2k+1)!(2k)!\pi)^{\frac{1}{2}}} \tag{2.2}$$

since the other term follows by interchanging x and y. The Laguerre polynomial of argument y in (2.2) has parameter $\alpha = -\frac{1}{2}$; thus, in order to apply Theorem 1.1, we write y in the form $y = (4k-1+2c')^{\frac{1}{2}} + \frac{Y}{2^{\frac{2}{3}}k^{\frac{1}{6}}}$ which corresponds to $c' = c+1$. Next we use Stirling's formula to estimate

$$\frac{2^{4k+1}(k!)^2 x}{(2^{4k+1}(2k+1)!(2k)!\pi)^{\frac{1}{2}}}(-1)^k 2^{-\frac{1}{2}-\frac{1}{3}} k^{-\frac{1}{3}} (-1)^k 2^{\frac{1}{2}-\frac{1}{3}} k^{-\frac{1}{3}}\,.$$

[3]For n odd the same analysis can be carried out to produce the same result.

The last factors in the left hand side are the constant factor in Theorem (1.1) for the Laguerre functions in x and y respectively. This expression is

$$2^{\frac{1}{3}}k^{-\frac{1}{6}}\left(1+\frac{X}{2^{\frac{5}{3}}k^{\frac{2}{3}}}+\frac{c}{4k}+O(k^{-\frac{5}{3}})\right).$$

This times, the constant[4] $\sqrt{\frac{n+1}{2}}$ from (2.1), gives

$$2^{\frac{1}{3}}k^{\frac{1}{3}}(1+\frac{X}{2^{\frac{5}{3}}k^{\frac{2}{3}}}+\frac{c+1}{4k}+O(k^{-\frac{5}{3}})).$$

We substitute all these into (2.2), and then interchange x and y to have the second term $\varphi_{n+1}(y)\varphi_n(x)$. Finally with the help of *Mathematica* we derive the following version of Theorem 1.2.

$$\text{For } x=(2n+1+2c)^{\frac{1}{2}}+\frac{X}{2^{\frac{1}{2}}n^{\frac{1}{6}}} \text{ and } y=(2n+1+2c)^{\frac{1}{2}}+\frac{Y}{2^{\frac{1}{2}}n^{\frac{1}{6}}}\,, \tag{2.3}$$

$$\begin{aligned} K_{n+1}(x,y)dx = \Bigg\{ & K_{\mathrm{Ai}}(X,Y)+\frac{1-2c}{2}\operatorname{Ai}(X)\operatorname{Ai}(Y)n^{-\frac{1}{3}}+\Bigg[\frac{X+Y}{20}\operatorname{Ai}'(X)\operatorname{Ai}'(Y)- \\ & \frac{X^2+XY+Y^2}{20}\operatorname{Ai}(X)\operatorname{Ai}(Y)+\frac{-10c^2+10c-1}{20}(\operatorname{Ai}'(X)\operatorname{Ai}(Y)+\operatorname{Ai}(X)\operatorname{Ai}'(Y))\Bigg]n^{-\frac{2}{3}} \\ & +O(\frac{1}{n})\ E(X,Y)\Bigg\}\,dX \end{aligned} \tag{2.4}$$

Using the symmetry between the X and Y terms from (2.1), this error term can be expressed as a finite sum

$$\begin{aligned} E(X,Y) \;=\; & \sum_j P_j(X,Y)\operatorname{Ai}(X)\operatorname{Ai}(Y)+\sum_{j_1}Q_{j_1}(X,Y)\operatorname{Ai}'(X)\operatorname{Ai}(Y)+ \\ & \sum_{j_2}Q_{j_2}(X,Y)\operatorname{Ai}(X)\operatorname{Ai}'(Y)+\sum_j R_j(X,Y)\operatorname{Ai}'(X)\operatorname{Ai}'(Y)+ \\ & \sum_{j,k}a_{j,k}\frac{X^jY^k\operatorname{Ai}(X)\operatorname{Ai}'(Y)-X^kY^j\operatorname{Ai}'(X)\operatorname{Ai}(Y)}{X-Y} \end{aligned}$$

where all coefficients of the polynomials P, Q and R, and the $a_{j,k}$ have a factor of $n^{-\frac{1}{3}k}$, $k\in\{0,1,2,3\}$.

The first four terms are kernels of finite rank operators on any Borel subset J of $\mathbb{R}$ not including minus infinity;[5] and thus, are trace class. The last term also defines

[4] Recall that in this section the notation K_n stands for an ensemble of $(n+1)\times(n+1)$ matrices.

[5] This last restriction is due to the behavior of the Airy function near minus infinity.

a trace class operator. This is best seen from the following result. If we assumed without lost of generalities that $j \le k$, and set $k - j = s$, then

$$\begin{aligned} & \frac{X^j Y^k \operatorname{Ai}(X) \operatorname{Ai}'(Y) - X^k Y^j \operatorname{Ai}'(X) \operatorname{Ai}(Y)}{X - Y} \\ = \;& (XY)^j \frac{(Y^s - X^s + X^s) \operatorname{Ai}(X) \operatorname{Ai}'(Y) - X^s \operatorname{Ai}'(X) \operatorname{Ai}(Y)}{X - Y} \\ = \;& (XY)^j \left(X^s K_{\mathrm{Ai}}(X, Y) \; + \; \sum_{i=0}^{s-1} Y^{s-i} X^i \operatorname{Ai}(X) \operatorname{Ai}'(Y) \right). \end{aligned}$$

This shows that the error $E(X, Y)$ in (2.4) is the kernel of a trace class operator.

2.2 Laguerre case

Again by the Christoffel-Darboux formula,

$$K_{n+1}^{\alpha}(x, y) = \frac{(n+1)(xy)^{\frac{\alpha}{2}}}{\Gamma(\alpha+1) \begin{pmatrix} n+\alpha \\ n \end{pmatrix}} \frac{\phi_{n+1}^{\alpha}(x)\phi_n^{\alpha}(y) - \phi_{n+1}^{\alpha}(y)\phi_n^{\alpha}(x)}{x - y} \,. \qquad (2.5)$$

2.2.1 Asymptotic of $e^{-\frac{x}{2}} L_n^{\alpha}(x)$ at $x = 4n + 2\alpha + 2c + 2(2n)^{\frac{1}{3}} X$

In order to apply Theorem 1.1 to the Laguerre kernel at the edge of the spectrum (corresponding to $x = 4n + 2\alpha + 2c + 2(2n)^{\frac{1}{3}} X$ for bounded c and X), we need to make a variable change $x = \xi^2$ where $\xi = \sqrt{4n + 2\alpha + 2c} + \; 2^{-\frac{2}{3}} n^{-\frac{1}{6}} \; t$. We use these two expressions to solve for t in terms of X and then substitute this value for t into Theorem 1.1 to obtain the desired asymptotics. But[6] in order to have accurate asymptotics for the Laguerre functions at the edge of the spectrum, we will use the expression of ξ involving $l_n = (4n + 2\alpha + 2c)^{1/2}$. Thus

$$x = \xi^2 \quad \text{is equivalent to} \quad l_n^2 + 2(2n)^{\frac{1}{3}} X = l_n^2 + (2l_n)^{\frac{2}{3}} t + (2l_n)^{-\frac{2}{3}} t^2 \,.$$

This quadratic has solutions $t_{\pm} = \dfrac{1}{2}(-(2l_n)^{\frac{4}{3}} \pm (2l_n)^{\frac{4}{3}} \sqrt{1 + 4(2l_n)^{-2} 2(2n)^{\frac{1}{3}} X}) \,.$

Actually, only the solution with the plus sign is to be taken as it is the only one bounded when n increases. An expansion of $t_+ = t$ leads to:

$$t = X - \frac{(\alpha + c)}{6n} X - \frac{X^2}{2^{\frac{8}{3}} n^{\frac{2}{3}}} + O(n^{-\frac{4}{3}}) \,.$$

Thus if in Theorem 1.1 we replace X by this value of t, we obtain, again with the help of *Mathematica*, the desired expansion.

[6] See the last footnote in the proof of Theorem 1.1 in the Appendix for this technical point.

Lemma 2.1. *For* $x = 4n + 2\alpha + 2c + 2(2n)^{\frac{1}{3}}X$ *and* X *bounded,*

$$e^{-\frac{x}{2}} L_n^\alpha(x) = (-1)^n 2^{-\alpha-\frac{1}{3}} n^{-\frac{1}{3}} \left\{ \mathrm{Ai}(X) + \frac{(c-1)}{2^{\frac{1}{3}}} \mathrm{Ai}'(X) n^{-\frac{1}{3}} + \right.$$

$$\left[\frac{2 - 10c + 5c^2 - 5\alpha}{10 \cdot 2^{\frac{2}{3}}} X \,\mathrm{Ai}(X) - \frac{2X^2}{10 \cdot 2^{\frac{2}{3}}} \mathrm{Ai}'(X) \right] n^{-\frac{2}{3}} +$$

$$\frac{1}{60} \left[- \left(6 + 56c + 15c^2 - 2c^3 + 5\alpha(3c-1) - 6X^3 + 6cX^3\right) \mathrm{Ai}(X) \right.$$

$$\left. \left. + \left(6 + \alpha(5 - 15c) - 6c - 15c^2 + 5c^3\right) X \,\mathrm{Ai}'(X) \right] n^{-1} \;+\; O(n^{-4/3})\; \mathrm{Ai}(X) \right\} \tag{2.6}$$

2.2.2 Asymptotic of $e^{-\frac{x}{2}} L_{n+1}^\alpha(x)$ at $x = 4n + 2\alpha + 2c + 2(2n)^{\frac{1}{3}}X$

Making use of this last formula, we can derive an asymptotic for $e^{-\frac{x}{2}} L_{n+1}^\alpha(x)$ when $x = 4n + 2\alpha + 2c + 2(2n)^{\frac{1}{3}}X$. Note that the degree of the Laguerre polynomial is no longer n but $n+1$, so in order to use Lemma2.1, we need to write x in terms of $n+1$ or $x = 4(n+1) + 2\alpha + 2(c-2) + 2(2(n+1))^{\frac{1}{3}}[X - \frac{1}{3n}X + O(n^{-2})]$. The substitution needed here is $c \to c-2$, and $X \to X - \frac{1}{3n}X$. *Mathematica* again gives:

Lemma 2.2. *For* $x = 4n + 2\alpha + 2c + 2(2n)^{\frac{1}{3}}X$ *and* X *bounded,*

$$e^{-\frac{x}{2}} L_{n+1}^\alpha(x) = (-1)^{n+1} 2^{-\alpha-\frac{1}{3}} (n+1)^{-\frac{1}{3}} \left\{ \mathrm{Ai}(X) + \frac{(c-3)}{2^{\frac{1}{3}}} \mathrm{Ai}'(X) n^{-\frac{1}{3}} + \right.$$

$$\left[\frac{42 - 30c + 5c^2 - 5\alpha}{10 \cdot 2^{\frac{2}{3}}} X \,\mathrm{Ai}(X) - \frac{2X^2}{10 \cdot 2^{\frac{2}{3}}} \mathrm{Ai}'(X) \right] n^{-\frac{2}{3}} +$$

$$\frac{1}{60} \left[\left(30 + 28c - 27c^2 + 2c^3 - 5\alpha(3c-7) + 18X^3 - 6cX^3\right) \mathrm{Ai}(X) \right.$$

$$\left. \left. + (-102 - 5\alpha(3c-7) + 114c - 45c^2 + 5c^3) X \,\mathrm{Ai}'(X) \right] n^{-1} \;+\; O(n^{-4/3})\; \mathrm{Ai}(X) \right\} \tag{2.7}$$

To complete this subsection we need to estimate

$$\frac{(n+1)(xy)^{\frac{\alpha}{2}}}{\Gamma(\alpha+1) \begin{pmatrix} n+\alpha \\ n \end{pmatrix}}.$$

We have

$$(xy)^{\frac{\alpha}{2}} = 2^{2\alpha} n^\alpha [1 + \frac{\alpha(X+Y)}{2^{\frac{5}{3}} n^{\frac{2}{3}}} + \frac{\alpha(\alpha+c)}{2n} + O(n^{-\frac{4}{3}})],$$

$$\Gamma(\alpha+1)\left(\begin{array}{c} n+\alpha \\ n \end{array}\right) = n^{\alpha}[1+\frac{\alpha^2+\alpha}{2n}+O(n^{-2})]$$

and in addition, the product of the constant factor on the right of (2.6) and (2.7) is

$$(-1)^{n+1}2^{-\alpha-\frac{1}{3}}(n+1)^{-\frac{1}{3}}\cdot(-1)^n 2^{-\alpha-\frac{1}{3}}n^{-\frac{1}{3}} = -2^{-2\alpha}n^{-\frac{2}{3}}[1-\frac{1}{3n}+O(n^{-2})]$$

thus

$$\frac{(n+1)(xy)^{\frac{\alpha}{2}}}{\Gamma(\alpha+1)\left(\begin{array}{c} n+\alpha \\ n \end{array}\right)}\cdot(-1)^{n+1}2^{-\alpha-\frac{1}{3}}(n+1)^{-\frac{1}{3}}\cdot(-1)^n 2^{-\alpha-\frac{1}{3}}n^{-\frac{1}{3}} =$$

$$-2^{-\frac{2}{3}}n^{\frac{1}{3}}[1+\frac{\alpha(X+Y)}{2^{\frac{5}{3}}n^{\frac{2}{3}}}+\frac{3\alpha(c-1)+4}{6n}+O(n^{-\frac{4}{3}})]$$

Substituting all these quantities in (2.5), give the following version of Theorem 1.3

$$\text{For } x = 4n+2\alpha+2c+2(2n)^{\frac{1}{3}}X \text{ and } y = 4n+2\alpha+2c+2(2n)^{\frac{1}{3}}Y \tag{2.8}$$

$$K^{\alpha}_{n+1}(x,y)dx = \Big\{K_{\mathrm{Ai}}(X,Y)+\frac{2-c}{2^{\frac{1}{3}}}\ \mathrm{Ai}(X)\ \mathrm{Ai}(Y)\, n^{-\frac{1}{3}}+$$

$$\frac{1}{2^{\frac{2}{3}}\,10}\Big[2(X^2+XY+Y^2)\ \mathrm{Ai}(X)\ \mathrm{Ai}(Y)-\ 2(X+Y)\ \mathrm{Ai}'(X)\ \mathrm{Ai}'(Y)-$$

$$(18-20c+5c^2)(\mathrm{Ai}(X)\ \mathrm{Ai}'(Y)+\mathrm{Ai}'(X)\ \mathrm{Ai}(Y))\Big]\, n^{-\frac{2}{3}}+\ O(\frac{1}{n})\ F(X,Y)\Big\}\, dX. \tag{2.9}$$

As in the Hermite case, the error term $F(X,Y)$ is a finite sum of kernels of trace class operators, therefore is a kernel of a trace class operator on $L^2(J)$ for any subset J of the reals which is bounded away from minus infinity.

2.3 Conclusion

For our order of expansion, we see that both kernels are finite rank perturbation of the Airy kernel. In the Laguerre case, the final result does not involve an explicit presence of the order α.

3 Expansion of the Fredholm determinants at the edge of the spectrum

This part will only make use (2.3) and (2.4) to derive the desired result in the Hermite case and (2.8) and (2.9) in the Laguerre case. Most of the derivations will follow from the work of Tracy and Widom.

3.1 Edgeworth expansion of $F_{n,2}^G$

Recall that

$$F_{n+1,2}^G(t) = \det(I - K_{n+1}) \tag{3.1}$$

where this determinant is the Fredholm determinant of the integral operator with kernel $K_{n+1}(x,y)$ on $L^2(s,\infty)$, t and s are related by

$$t = (2\,n + 1 + 2\,c)^{\frac{1}{2}} + 2^{-\frac{1}{2}}\,n^{-\frac{1}{6}}\,s\,. \tag{3.2}$$

In this section we will estimate this determinant. Most of our derivations involve trace class operators where the Fredholm determinant is analytic. If in (3.1) we use the expression of K_{n+1} given by (2.4), the continuity of the determinant (in trace class norm) allows us to pull the error term involving the kernel $E(X,Y)$ out of the determinant as an $O(n^{-1})$ term. We therefore have

$$\det(I - K_{n+1}) = \det\Bigg(I - \bigg\{K_{\mathrm{Ai}}(X,Y) + \frac{1-2c}{2}\,\mathrm{Ai}(X)\,\mathrm{Ai}(Y) n^{-\frac{1}{3}} +$$

$$\bigg[\frac{X+Y}{20}\,\mathrm{Ai}'(X)\,\mathrm{Ai}'(Y) - \frac{X^2+XY+Y^2}{20}\,\mathrm{Ai}(X)\,\mathrm{Ai}(Y) +$$

$$\frac{-10c^2+10c-1}{20}(\mathrm{Ai}'(X)\,\mathrm{Ai}(Y) + \mathrm{Ai}(X)\,\mathrm{Ai}'(Y))\bigg]\,n^{-\frac{2}{3}}\bigg\}\Bigg) + O(n^{-1}). \tag{3.3}$$

Note we are using the obvious notation of writing the kernel for the operator appearing in the determinant. We continue to employ this notation below. If we factor out $\det(I - K_{\mathrm{Ai}}(X,Y))$ and set

$$K_{n+1}(x,y)\,dx - K_{\mathrm{Ai}}(X,Y)\,dX := L(X,Y)\,dX + O(n^{-1})E(X,Y)\,dX,$$

the multiplicative property of the determinant gives

$$\det(I - K_{n+1}(x,y)) = \det(I - K_{\mathrm{Ai}}(X,Y))\det(I - (I - K_{\mathrm{Ai}}(X,Y))^{-1}L(X,Y)) + O(n^{-1}). \tag{3.4}$$

The first factor on the right is known, so we need to estimate the second factor. We will express this factor as a finite sum of rank one operators

$$(I - K_{\mathrm{Ai}}(X,Y))^{-1}L(X,Y) = \sum_{i=1}^{k}\alpha_i(X)\beta_i(Y) \tag{3.5}$$

and use the well known formula

$$\det\left(I - \sum_{i=1}^{k}\alpha_i(X)\beta_i(Y)\right) = \det\left(\delta_{ij} - (\alpha_i,\beta_j)\right)_{1\le i,j\le k} \tag{3.6}$$

to transform the problem to one involving estimations of inner products. Let $(\cdot,\cdot)$ denote the inner product in $L^2(s,\infty)$. A good reference for much of the results that follows is [20]. We have the following representation.

$$(I-K_{\mathrm{Ai}}(X,Y))^{-1}L(X,Y)=\frac{1-2c}{2}(I-K_{\mathrm{Ai}}(X,Y))^{-1}\operatorname{Ai}(X)\operatorname{Ai}(Y)n^{-\frac{1}{3}}+$$

$$\frac{1}{20}\Big[(I-K_{\mathrm{Ai}}(X,Y))^{-1}X\operatorname{Ai}'(X)\operatorname{Ai}'(Y)+(I-K_{\mathrm{Ai}}(X,Y))^{-1}Y\operatorname{Ai}'(X)\operatorname{Ai}'(Y)-$$

$$(I-K_{\mathrm{Ai}}(X,Y))^{-1}X^2\operatorname{Ai}(X)\operatorname{Ai}(Y)-(I-K_{\mathrm{Ai}}(X,Y))^{-1}XY\operatorname{Ai}(X)\operatorname{Ai}(Y)-$$

$$(I-K_{\mathrm{Ai}}(X,Y))^{-1}Y^2\operatorname{Ai}(X)\operatorname{Ai}(Y)+(-10c^2+10c-1)(I-K_{\mathrm{Ai}}(X,Y))^{-1}\operatorname{Ai}'(X)\operatorname{Ai}(Y)$$

$$+(-10c^2+10c-1)(I-K_{\mathrm{Ai}}(X,Y))^{-1}\operatorname{Ai}(X)\operatorname{Ai}'(Y)\Big]\,n^{-\frac{2}{3}}\;+\;O(n^{-1})$$

In the notations above, we think of all the quantities involved as kernels of integral operators, the analogues of those in [20]. We therefore have:

$$\frac{1-2c}{2}(I-K_{\mathrm{Ai}}(X,Y))^{-1}\operatorname{Ai}(X)\operatorname{Ai}(Y)n^{-\frac{1}{3}}=\frac{1-2c}{2}Q(X)\operatorname{Ai}(Y)n^{-\frac{1}{3}}:=\alpha_1(X)\beta_1(Y)$$

where $Q(X)$ is the action of the integral operator with kernel $(I-K_{\mathrm{Ai}})^{-1}$ on $\operatorname{Ai}(X)$. In the same way we have,

$$\frac{1}{20}(I-K_{\mathrm{Ai}}(X,Y))^{-1}X\operatorname{Ai}'(X)\operatorname{Ai}'(Y)n^{-\frac{2}{3}}=\frac{1}{20}((I-K_{\mathrm{Ai}})^{-1}f)(X)\operatorname{Ai}'(Y)n^{-\frac{2}{3}}$$
$$:=\alpha_2(X)\beta_2(Y)$$

where $f(Z)=Z\operatorname{Ai}'(Z)$ and,

$$\frac{1}{20}(I-K_{\mathrm{Ai}}(X,Y))^{-1}\operatorname{Ai}'(X)Y\operatorname{Ai}'(Y)n^{-\frac{2}{3}}=\frac{1}{20}P(X)f(Y)n^{-\frac{2}{3}}$$
$$:=\alpha_3(X)\beta_3(Y)$$

where $P(X)$ is the action of the integral operator with kernel $(I-K_{\mathrm{Ai}})^{-1}$ acting on $\operatorname{Ai}'(X)$.

$$-\frac{1}{20}(I-K_{\mathrm{Ai}}(X,Y))^{-1}X^2\operatorname{Ai}(X)\operatorname{Ai}(Y)n^{-\frac{2}{3}}=-\frac{1}{20}((I-K_{Airy})^{-1}g)(x)\operatorname{Ai}(y)n^{-\frac{2}{3}}$$
$$:=\alpha_4(X)\beta_4(Y)$$

where $g(Z)=Z^2\operatorname{Ai}(Z)$,

$$-\frac{1}{20}(I-K_{\mathrm{Ai}}(X,Y))^{-1}XY\operatorname{Ai}(X)\operatorname{Ai}(Y)n^{-\frac{2}{3}}=-\frac{1}{20}((I-K_{\mathrm{Ai}})^{-1}h)(X)h(Y)n^{-\frac{2}{3}}$$
$$:=\alpha_5(X)\beta_5(Y)$$

where $h(Z) = Z\,\mathrm{Ai}(Z)$,

$$-\frac{1}{20}(I - K_{\mathrm{Ai}}(X,Y))^{-1}\,\mathrm{Ai}(X)Y^2\,\mathrm{Ai}(Y)n^{-\frac{2}{3}} = -\frac{1}{20}Q(X)g(Y)n^{-\frac{2}{3}} := \alpha_6(X)\beta_6(Y)$$

$$C\,(I - K_{\mathrm{Ai}}(X,Y))^{-1}\,\mathrm{Ai}'(X)\,\mathrm{Ai}(Y)n^{-\frac{2}{3}} = CP(X)\,\mathrm{Ai}(Y)n^{-\frac{2}{3}} := \alpha_7(X)\beta_7(Y)\,, \quad \text{and}$$

$$C\,(I - K_{\mathrm{Ai}}(X,Y))^{-1}\,\mathrm{Ai}(X)\,\mathrm{Ai}'(Y)n^{-\frac{2}{3}} = CQ(X)\,\mathrm{Ai}'(Y)n^{-\frac{2}{3}} := \alpha_8(X)\beta_8(Y)$$

where $C = (-10c^2 + 10c - 1)/20$.

If we set $(\alpha_i, \beta_j) = a_{ij}n^{-\frac{2}{3}}$ for $j \neq 1$ and $(\alpha_i, \beta_1) = a_{i1}n^{-\frac{1}{3}}$, expanding (3.6) with respect to the first row leads to the following expression.

$$\begin{aligned}
\det(\delta_{ij} - (\alpha_i, \beta_j))_{i,j=1}^8 &= (1 - a_{11}n^{-\frac{1}{3}})\begin{vmatrix} 1 - a_{22}n^{-\frac{2}{3}} & -a_{23}n^{-\frac{2}{3}} & \cdots & -a_{28}n^{-\frac{2}{3}} \\ -a_{32}n^{-\frac{2}{3}} & 1 - a_{33}n^{-\frac{2}{3}} & \cdots & -a_{38}n^{-\frac{2}{3}} \\ \cdot & \cdot & \cdot & \cdot \\ -a_{82}n^{-\frac{2}{3}} & -a_{83}n^{-\frac{2}{3}} & \cdots & 1 - a_{88}n^{-\frac{2}{3}} \end{vmatrix} \\
&+ \sum_{k=2}^{8}(-1)^k a_{1k}n^{-\frac{2}{3}}\det(C_1, \cdots, \hat{C}_k, \cdots, C_8) \\
&= (1 - a_{11}n^{-\frac{1}{3}})\begin{vmatrix} 1 - a_{22}n^{-\frac{2}{3}} & -a_{23}n^{-\frac{2}{3}} & \cdots & -a_{28}n^{-\frac{2}{3}} \\ -a_{32}n^{-\frac{2}{3}} & 1 - a_{33}n^{-\frac{2}{3}} & \cdots & -a_{38}n^{-\frac{2}{3}} \\ \cdot & \cdot & \cdot & \cdot \\ -a_{82}n^{-\frac{2}{3}} & -a_{83}n^{-\frac{2}{3}} & \cdots & 1 - a_{88}n^{-\frac{2}{3}} \end{vmatrix} \\
&+ O(n^{-1}).
\end{aligned}$$

We factor out $n^{-\frac{1}{3}}$ from column C_1 in the last step. The determinant in the last line is of the same form as the the original determinant, therefore a similar transformation to this last determinant leads to the following result.

$$\begin{aligned}
\det(\delta_{ij} &- (\alpha_i, \beta_j))_{1\le i,j\le 8} = (1 - a_{11}n^{-\frac{1}{3}})\prod_{k=2}^{8}(1 - a_{kk}n^{-\frac{2}{3}}) + O(n^{-1}) \\
&= (1 - a_{11}n^{-\frac{1}{3}})\sum_{k=0}^{7}(-1)^k n^{-\frac{2k}{3}} \sum_{i_1,\cdots,i_k; i_r\neq i_s\in\{2,\ldots,8\}} \prod_{j=1}^{k} a_{i_j i_j} + O(n^{-1}) \\
&= 1 - a_{11}n^{-\frac{1}{3}} - n^{-\frac{2}{3}}\sum_{k=2}^{8} a_{kk} + O(n^{-1}) = 1 - \sum_{k=1}^{8}(\alpha_k, \beta_k) + O(n^{-1}).
\end{aligned}$$

Thus we only need to compute the inner products (α_k, β_k) for $k = 1, \cdots, 8$.
To simplify notations we will write for example u_0 instead of $u_0(s)$. We therefore have:

$$(\alpha_1, \beta_1) = \frac{1-2c}{2}(Q, \mathrm{Ai})n^{-\frac{1}{3}} = \frac{1-2c}{2}\,u_0\,n^{-\frac{1}{3}}$$

$$(\alpha_2,\beta_2) \;=\; \frac{n^{-\frac{2}{3}}}{20}((I-K_{\mathrm{Ai}})^{-1}X\,\mathrm{Ai}', \mathrm{Ai}') = \frac{n^{-\frac{2}{3}}}{20}(P, X\,\mathrm{Ai}') = \frac{n^{-\frac{2}{3}}}{20}w_1$$

$$(\alpha_3,\beta_3) \;=\; \frac{n^{-\frac{2}{3}}}{20}(P, X\,\mathrm{Ai}') = \frac{n^{-\frac{2}{3}}}{20}w_1$$

$$(\alpha_4,\beta_4) \;=\; -\frac{n^{-\frac{2}{3}}}{20}((I-K_{\mathrm{Ai}})^{-1}g, \mathrm{Ai}) = -\frac{n^{-\frac{2}{3}}}{20}(Q, X^2\,\mathrm{Ai}) = -\frac{n^{-\frac{2}{3}}}{20}u_2$$

$$(\alpha_5,\beta_5) \;=\; -\frac{n^{-\frac{2}{3}}}{20}((I-K_{\mathrm{Ai}})^{-1}h, h) = -\frac{n^{-\frac{2}{3}}}{20}(Q_1, X\,\mathrm{Ai})$$

$$(\alpha_6,\beta_6) \;=\; -\frac{n^{-\frac{2}{3}}}{20}(Q, X^2\,\mathrm{Ai}) = -\frac{n^{-\frac{2}{3}}}{20}u_2$$

$$(\alpha_7,\beta_7) \;=\; Cn^{-\frac{2}{3}}(P, \mathrm{Ai}) = Cn^{-\frac{2}{3}}(Q, \mathrm{Ai}') = Cn^{-\frac{2}{3}}v_0 \quad = \quad (\alpha_8,\beta_8).$$

To estimate (α_5,β_5), we use equation 2.12 of [20] which says

$$Q_1(X) = XQ(X) + u_0P(X) - v_0Q(X), \quad \text{to have}$$

$$\begin{aligned}
(Q_1(X), X\,\mathrm{Ai}(X)) &= (X\,Q(X) + u_0P(X) - v_0Q(X), X\,\mathrm{Ai}(X)) \\
&= u_2 - v_0u_1 + u_0(P(X), X\,\mathrm{Ai}(X)) \\
&= u_2 - v_0u_1 + u_0\widetilde{v}_1 \\
&= u_2 - v_0u_1 + u_0v_1 - u_0v_0^2 + u_0^2w_0. \quad (\text{ We used } (2.14) \text{ of } [20])
\end{aligned}$$

And

$$(\alpha_5,\beta_5) = -\frac{n^{-\frac{2}{3}}}{20}(u_2 - u_1v_0 + u_0v_1 - u_0v_0^2 + u_0^2w_0).$$

Substituting this into the formula for the determinant gives

$$\det(\delta_{ij} - (\alpha_i,\beta_j))_{1\le i,j\le 8} = 1 - \sum_{k=1}^{8}(\alpha_k,\beta_k) + O(n^{-1})$$

$$\begin{aligned}
= \;& 1 - \frac{1-2c}{2}u_0n^{-\frac{1}{3}} - \frac{n^{-\frac{2}{3}}}{20}\left\{2w_1 - 3u_2 + (-20c^2+20c-2)v_0 + u_1v_0\right. \\
& \left. - u_0v_1 + u_0v_0^2 - u_0^2w_0\right\} \quad + \quad O(n^{-1}).
\end{aligned}$$

If we set

$$a^G_{c,2}(s) = \frac{2c-1}{2}\;, \quad b^G_2 = -\frac{1}{20} \quad \text{and}$$

$$E^G_{c,2}(s) = 2w_1 - 3u_2 + (-20c^2+20c-2)v_0 + u_1v_0 - u_0v_1 + u_0v_0^2 - u_0^2w_0, \qquad (3.7)$$

equation (3.4) gives

Lemma 3.1. *For* $t_c = (2n+1+2c)^{\frac{1}{2}} + 2^{-\frac{1}{2}}\, n^{-\frac{1}{6}}\, s$*,*

$$F^G_{n+1,2}(t_c) = F_2(s)\left\{1 + a^G_{c,2}\, u_0(s)\, n^{-\frac{1}{3}} + b^G_2\, E^G_{c,2}(s)\, n^{-\frac{2}{3}}\right\} + O(n^{-1}) \tag{3.8}$$

For $c = \frac{1}{2}$ we have faster convergence in Lemma 3.1 since $a^G_{\frac{1}{2},2} = 0$.

3.2 Edgeworth Expansion of $F^L_{n,2}$

From the similarities between the two expressions in (2.4) and (2.9), the derivation of $F^L_{n+1,2}$ follows exactly the same steps as the previous one, they would differ only by some constant terms.

We see that the corresponding inner products in terms of the α's are

$$\begin{aligned}
(\alpha_1,\beta_1) &= \frac{2-c}{2^{\frac{1}{3}}} u_0 n^{-\frac{1}{3}} \quad , \quad (\alpha_5,\beta_5) = -\frac{n^{-\frac{2}{3}}}{2^{\frac{2}{3}}\cdot 10} 2w_1 \quad , \quad (\alpha_6,\beta_6) = -\frac{n^{-\frac{2}{3}}}{2^{\frac{2}{3}}\cdot 10} 2w_1,\\
(\alpha_2,\beta_2) &= \frac{n^{-\frac{2}{3}}}{2^{\frac{2}{3}}\cdot 10} 2u_2 \quad , \quad (\alpha_3,\beta_3) = \frac{n^{-\frac{2}{3}}}{2^{\frac{2}{3}}\cdot 10} 2(u_2 - u_1 v_0 + u_0 v_1 - u_0 v_0^2 + u_0^2 w_0),\\
(\alpha_4,\beta_4) &= \frac{n^{-\frac{2}{3}}}{2^{\frac{2}{3}}\cdot 10} 2u_2 \quad , \quad (\alpha_7,\beta_7) = -\frac{n^{-\frac{2}{3}}}{2^{\frac{2}{3}}\cdot 10} (18 - 20c + 5c^2) v_0 = (\alpha_8,\beta_8).
\end{aligned}$$

If we set

$$a^L_{c,2}(s) = \frac{c-2}{2^{\frac{1}{3}}}\,, \quad b^L_2 = \frac{2^{\frac{1}{3}}}{10} \quad \text{and}$$

$$E^L_{c,2}(s) = 2w_1 - 3u_2 + (5c^2 - 20c + 18)v_0 + u_1 v_0 - u_0 v_1 + u_0 v_0^2 - u_0^2 w_0, \tag{3.9}$$

our formula reads

Lemma 3.2. *For* $t_c = 4n + 2\alpha + 2c + 2(2n)^{\frac{1}{3}} s$*,*

$$F^L_{n+1,2}(t_c) = F_2(s)\{1 + a^L_{c,2}\, u_0(s)\, n^{-\frac{1}{3}} + b^L_2\, E^L_{c,2}(s)\, n^{-\frac{2}{3}}\} \quad + \quad O(n^{-1}). \tag{3.10}$$

For $c = 2$ we obtain a faster convergence as $a^L_{2,2} = 0$.

3.3 Fine tuning

To complete this analysis, we need to find values for the constant $c = c_1$ in Lemma 3.1 and $c = c_2$ in Lemma 3.2 for which $E^G_{c_1,2}(s) = E^L_{c_2,2}(s)$. Which is equivalent to

$$-20c_1^2 + 20c_1 - 2 = 5c_2^2 - 20c_2 + 18 \quad \Leftrightarrow \quad (c_1 - \frac{1}{2})^2 + \frac{1}{4}(c_2 - 2)^2 = \frac{1}{4} \tag{3.11}$$

This suggest the following change of variables.

$$c_G = c_1 - \frac{1}{2} \quad \text{and} \quad c_L = \frac{c_2 - 2}{2}. \quad \text{Therefore}$$

$$(3.11) \quad \text{changes to} \quad c_G^2 + c_L^2 = \frac{1}{4}\,, \quad a^G_{c_G,2} = c_G\,, \quad a^L_{c_L,2} = 2^{\frac{2}{3}} c_L\,,$$
$$-20c_1^2 + 20c_1 - 2 \;=\; -20c_G^2 + 3\,, \quad 5c_2^2 - 20c_2 + 18 \;=\; 20c_L^2 - 2\,,$$

$$t_{c_G} = (2(n+1)+2c_G)^{\frac{1}{2}} + 2^{-\frac{1}{2}} n^{-\frac{1}{6}} s \quad \text{and} \quad t_{c_L} = 4(n+1)+2\alpha+4c_L+2(2n)^{\frac{1}{3}} s\,. \tag{3.12}$$

We can now give a scaling of t in terms of the size of the matrices.

$$t_{c_G} = (2(n+1)+2c_G)^{\frac{1}{2}} + 2^{-\frac{1}{2}}(n+1)^{-\frac{1}{6}} s \quad \text{for the Gaussian case, and}$$

$$t_{c_L} = 4(n+1)+2\alpha+4c_L+2(2(n+1))^{\frac{1}{3}} s \quad \text{for the Laguerre case.}$$

Since all the functions derived so far are all differentiable, this new scaling will only change the error function but not its order and class. To keep the notations light, we will use the same variable to represent these error functions. We therefore have Theorem 1.2 and Theorem 1.3 for an ensemble of $(n+1)\times(n+1)$ matrices.

Theorem 3.3. *For* $x = (2(n+1)+2c_G)^{\frac{1}{2}} + 2^{-\frac{1}{2}}(n+1)^{-\frac{1}{6}} X$ *and* $y = (2(n+1)+2c_G)^{\frac{1}{2}} + 2^{-\frac{1}{2}}(n+1)^{-\frac{1}{6}} Y$

$$K_{n+1}(x,y)\,dx = \Big\{ K_{\mathrm{Ai}}(X,Y) - c_G \operatorname{Ai}(X)\operatorname{Ai}(Y)(n+1)^{-\frac{1}{3}} +$$

$$\frac{1}{20}\Big[(X+Y)\operatorname{Ai}'(X)\operatorname{Ai}'(Y) - (X^2+XY+Y^2)\operatorname{Ai}(X)\operatorname{Ai}(Y) +$$

$$\frac{-20c_G^2+3}{2}(\operatorname{Ai}'(X)\operatorname{Ai}(Y) + \operatorname{Ai}(X)\operatorname{Ai}'(Y))\Big](n+1)^{-\frac{2}{3}} + O((n+1)^{-1})E(X,Y)\Big\}\,dX \tag{3.13}$$

The error term, $E(X,Y)$, is again a kernel of an integral operator on $L^2(J)$ which is trace class for any subset J of the reals that is bounded away from minus infinity.

Taking the limit as $Y \to X$ in (3.13) give the one point correlation function ρ_{n+1}.

Corollary 3.4.

$$\textit{For} \quad x = (2(n+c_G))^{\frac{1}{2}} + \frac{X}{2^{\frac{1}{2}} n^{\frac{1}{6}}},$$

$$2^{-\frac{1}{2}} n^{-\frac{1}{6}} \rho_n(x) = 2^{-\frac{1}{2}} n^{-\frac{1}{6}} K_n(x,x) = [\operatorname{Ai}'(X)]^2 - X[\operatorname{Ai}(X)]^2 - c_G\,[\operatorname{Ai}(X)]^2 n^{-\frac{1}{3}} +$$

$$\frac{1}{20}\Big\{2X[\operatorname{Ai}'(X)]^2 - 3X^2[\operatorname{Ai}(X)]^2 + (3-20c_G^2)\operatorname{Ai}'(X)\operatorname{Ai}(X)\Big\} n^{-\frac{2}{3}} + O(n^{-1})F_n(X) \tag{3.14}$$

Note that for $c_G = 0$, this is formula (72) of [7]. Figure 3.3 illustrates the accuracy of equation (3.14).

For the Laguerre case,

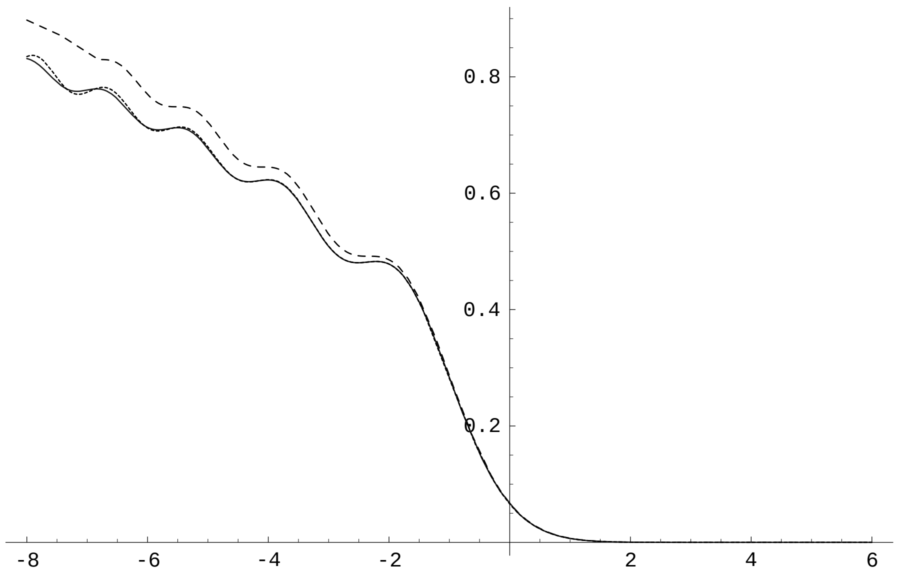

Figure 3.3.1: For $c_G = 0$ and $n = 40$, The dashed curve is the usual approximation of the one point correlation function from the Airy kernel (which is the first two terms in (3.14)), the solid curve is the exact scaled one point correlation function and the doted curve is our approximation (3.14)

Theorem 3.5. *For* $x = 4(n+1) + 2\alpha + 4c_L + 2(2(n+1))^{\frac{1}{3}}X$ *and* $y = 4(n+1) + 2\alpha + 4c_L + 2(2(n+1))^{\frac{1}{3}}Y$ *with* X, c_L *and* Y *bounded,*

$$K^{\alpha}_{n+1}(x,y)\,dx = \Big\{K_{\mathrm{Ai}}(X,Y) - 2^{\frac{2}{3}}c_L\,\mathrm{Ai}(X)\,\mathrm{Ai}(Y)(n+1)^{-\frac{1}{3}} +$$

$$\frac{2^{\frac{1}{3}}}{10}\Big[\quad (X^2+XY+Y^2)\,\mathrm{Ai}(X)\,\mathrm{Ai}(Y) - (X+Y)\,\mathrm{Ai}'(X)\,\mathrm{Ai}'(Y) -$$

$$(10c_L^2-1)(\mathrm{Ai}(X)\,\mathrm{Ai}'(Y) + \mathrm{Ai}'(X)\,\mathrm{Ai}(Y))\Big]\,(n+1)^{-\frac{2}{3}} + O((n+1)^{-1})F(X,Y)\Big\}\,dX \tag{3.15}$$

The error term $F(X,Y)$ is again the kernel of an integral operator on $L^2(J)$ which is trace class for any subset J of the reals which is bounded away from minus infinity.

Taking the limit as y goes to x in (3.15) gives the one point correlation function ρ^{α}_{n+1} in the Laguerre case.

Corollary 3.6. *For* $x = 4(n+c_L) + 2\alpha + 2(2n)^{\frac{1}{3}}X$,

$$2(2n)^{\frac{1}{3}}\rho^{\alpha}_n(x) = [\mathrm{Ai}'(X)]^2 - X[\mathrm{Ai}(X)]^2 - 2^{\frac{2}{3}}\,c_L[\mathrm{Ai}(X)]^2 n^{-\frac{1}{3}} +$$

$$\frac{2^{\frac{1}{3}}}{10}[3X^2[\mathrm{Ai}(X)]^2 - 2X[\mathrm{Ai}'(X)]^2 + 2(1-10\,c_L^2)\,\mathrm{Ai}(X)\,\mathrm{Ai}'(X)]n^{-\frac{2}{3}} + O(n^{-1})F(X) \tag{3.16}$$

Note that for $c_L = -\alpha/2$ this is formula (73) in [7]. Figure 3.3.2 illustrates the accuracy of our result. Combining Lemma 3.1, Lemma 3.2 and the fine-tuned constants in (3.12) give Theorem 1.4.

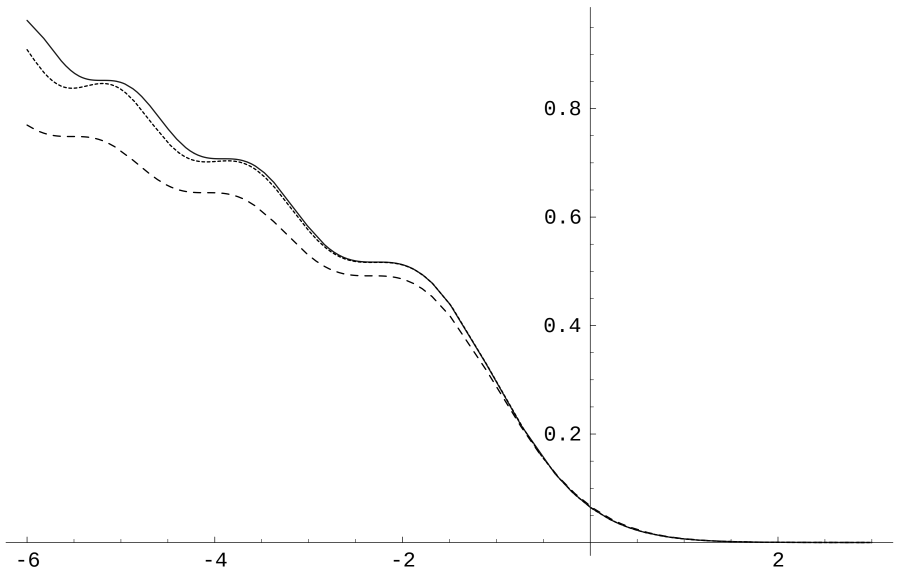

Figure 3.3.2: For $c_L = 0$, $\alpha = \frac{1}{2}$ and $n = 40$, the solid curve is the scaled one point correlation function, the doted curve is our approximation from (3.16), and the dashed one is the first order approximation from the Airy kernel.

A Proof of Theorem1.1

In this section we will follow Szegö [18], Section 8.75. With the following changes.

- We introduce a new variable c in the definition of l_n to fine-tune the final result. $l_n = (4n + 2\alpha + 2c)^{\frac{1}{2}}$ instead of $l_n = (4n + 2\alpha + 2)^{\frac{1}{2}}$.
- We will use the second order Hankel expansion of the Bessel function of large argument instead of the first order.
- We define $\xi = l_n + (2l_n)^{-\frac{1}{3}}t$ instead of $\xi = l_n - (6l_n)^{-\frac{1}{3}}t$ to match the definition of the Airy function with the one commonly used.
- We give an estimate of the error term as a function of the independent variable

recalling the generating function of the Laguerre polynomials.

$$\sum_{n=0}^{\infty} \frac{L_n^\alpha(x)}{\Gamma(n+\alpha+1)}\omega^n = e^\omega (x\omega)^{-\frac{\alpha}{2}} J_\alpha(2(x\omega)^{\frac{1}{2}}) \tag{A.1}$$

The substitutions $x \to \xi$ and $\omega \to -\frac{\omega^2}{4}$, and for edge scaling $l_n = (4n + 2\alpha + 2c)^{\frac{1}{2}}$, $\omega \to l_n z$. we deduce from (A.1)

$$\frac{L_n^\alpha(\xi^2)}{\Gamma(n+\alpha+1)}(-\frac{1}{4})^n = \frac{2^\alpha(\xi)^{-\alpha}}{2\pi i} l_n^{-2n-\alpha} \int_\gamma e^{-\frac{l_n^2 z^2}{4} - \frac{1}{2}l_n^2 log(z)} z^{c-1} e^{\frac{\pi\alpha i}{2}} J_\alpha(e^{-\frac{\pi i}{2}}\xi l_n z)dz \tag{A.2}$$

where γ is a symmetric contour enclosing the origin.

Using the Hankel expansion of the Bessel function of large argument (see, for example, Olver [15, pgs 130–132]): that[7] as $|l_n z| \to \infty$ in $|\arg(z)| \leq \pi - \delta(< \pi)$,

$$\begin{aligned} e^{\frac{\pi\alpha i}{2}} J_\alpha(e^{-\frac{\pi i}{2}}\xi l_n z) &= (2\pi\xi l_n)^{-\frac{1}{2}} z^{-\frac{1}{2}} \Big[e^{\xi l_n z} + e^{-\xi l_n z + (\alpha+\frac{1}{2})\pi i} \\ &+ \frac{4\alpha^2-1}{8\xi l_n z}(-e^{\xi l_n z} + e^{-\xi l_n z+(\alpha+\frac{1}{2})\pi i}) + O(\frac{1}{|\xi l_n z|^2}) \Big] \end{aligned}$$

Substituting this in (A.2) for $\xi = l_n + (2l_n)^{-\frac{1}{3}}t$ with t bounded,

$$\frac{L_n^\alpha(\xi^2)}{\Gamma(n+\alpha+1)}(-\frac{1}{4})^n = \frac{2^\alpha(l_n + (2l_n)^{-\frac{1}{3}}t)^{-\alpha}}{2\pi i} l_n^{-2n-\alpha}(2\pi(l_n + (2l_n)^{-\frac{1}{3}}t)l_n)^{-\frac{1}{2}}.$$

[7] In the following formula the O-term is actually of the form

$$O(\frac{1}{|\xi l_n z|^2})e^{\xi l_n z} + O(\frac{1}{|\xi l_n z|^2})e^{-\xi l_n z+(\alpha+\frac{1}{2})\pi i}.$$

$$\Big\{\int_\gamma (1-\frac{4\alpha^2-1}{8l_n^2 z(1+2^{-\frac{1}{3}}l_n^{-\frac{4}{3}}t)})e^{-\frac{l_n^2z^2}{4}-\frac{1}{2}l_n^2\log(z)+l_n^2z+2^{-\frac{1}{3}}l_n^{-\frac{1}{3}}l_nzt} \quad z^{c-\frac{3}{2}}dz \quad +$$

$$e^{(\alpha+\frac{1}{2})\pi i}\int_\gamma (1+\frac{4\alpha^2-1}{8l_n^2 z(1+2^{-\frac{1}{3}}l_n^{-\frac{4}{3}}t)})e^{-\frac{l_n^2z^2}{4}-\frac{1}{2}l_n^2\log(z)-l_n^2z-2^{-\frac{1}{3}}l_n^{-\frac{1}{3}}l_nzt} \quad z^{c-\frac{3}{2}}dz$$

$$+\,0(l_n^{-4})\int_\gamma |e^{-\frac{l_n^2z^2}{4}-\frac{1}{2}l_n^2\log(z)\pm l_n^2z\pm 2^{-\frac{1}{3}}l_n^{-\frac{1}{3}}l_nzt}| \quad |dz| \Big\} \tag{A.3}$$

The $\pm$ signs would be taken according to which one gives the larger contribution. To simplify notations we set $p=(4\alpha^2-1)/(8l_n^2z(1+2^{-\frac{1}{3}}l_n^{-\frac{4}{3}}t))$.

The integral over γ can be split into the path in the upper half-plane and the path in the lower half-plane. However the lower-half plane part of the path can be transformed into the upper half plane via the transformation $z\to\bar{z}$, taking into account the orientation change and the fact that the integrand is an analytic function of z on γ. The integrand will be transformed into its complex conjugate under this transformation. The Jacobian of the transformation is -1 so it will reverse the orientation once again. Thus the lower half plane contribution of the integral is equal to the conjugate of the upper half contribution such that the integral over γ is exactly twice the real part of the integral over the upper half portion γ_+ of the contour. Thus (A.3) is equal to

$$2^\alpha(l_n+(2l_n)^{-\frac{1}{3}}t)^{-\alpha}l_n^{-2n-\alpha}(2\pi(l_n+(2l_n)^{-\frac{1}{3}}t)l_n)^{-\frac{1}{2}}\cdot 2\mathcal{R}e\{\frac{1}{2\pi i}G+\frac{1}{2\pi i}H+K\} \tag{A.4}$$

where

$$G=\int_{\gamma_+}(1-p)e^{-l_n^2f_1(z)+2^{-\frac{1}{3}}l_n^{\frac{2}{3}}z\,t}\;z^{c-\frac{3}{2}}\,dz=\int_{\gamma_+}e^{-l_n^2f_1(z)}g(z)\,dz \tag{A.5}$$

$$\text{with}\quad f_1(z)=\frac{z^2}{4}-z+\frac{1}{2}\log(z)$$

$$H=e^{(\alpha+\frac{1}{2})\pi i}\int_{\gamma_+}(1+p)e^{-l_n^2f_2(z)-2^{-\frac{1}{3}}l_n^{\frac{2}{3}}zt}\;z^{c-\frac{3}{2}}\,dz=\int_{\gamma_+}e^{-l_n^2f_2(z)}h(z)\,dz \tag{A.6}$$

$$\text{with}\quad f_2(z)=\frac{z^2}{4}+z+\frac{1}{2}\log(z)$$

$$\text{and}\quad K=0(l_n^{-4})\int_{\gamma_+}|e^{-\frac{l_n^2z^2}{4}-\frac{1}{2}l_n^2log(z)\pm l_n^2z\pm 2^{-\frac{1}{3}}l_n^{\frac{2}{3}}zt}|\quad |dz| \tag{A.7}$$

We will use the steepest descent method to find an asymptotic expansion for G,H and K for large l_n keeping α fixe and c bounded.[8]

[8]Actually c can grow with n, but we are not going to look at this problem.

A.1 Steepest descent method for G

The steepest descent for this section is with respect to large l_n in (A.5). The saddle point condition $f^{'}(z) = 0$ gives $z_0 = 1$ with $f^{''}(1) = 0$.

The steepest descent curve leaves $z_0 = 1$ at angles $0,\ \pi/3,$ or $2\pi/3$. The direction of maximum decreases of f_1 is $2\pi/3$. We deform γ_+ at $z_0 = 1$ such that the resulting contour leaves $z_0 = 1$ at angle $2\pi/3$ as the line segment

$$z = 1 + 2^{\frac{1}{3}} l_n^{-\frac{2}{3}} \rho e^{\frac{2\pi i}{3}} \quad \text{with } 0 \le \rho \le n^{\delta} \quad \text{where} \quad 0 < \delta < \frac{1}{6} \tag{A.8}$$

then along the segment symmetric to this segment with respect to the imaginary axis and finally connect the tip of these two line segments by an arc of circle centered at the origin with radius r. For simplicity, we will also call this path[9] γ_+.

In the following sections, we will estimate the contribution of each portion of the contour to G.

A.1.1 On the arc of circle $z = re^{\phi i}$

Since $\mathcal{R}e(-f_1(z)) = -\frac{1}{4}r^2 \cos 2\phi + r\cos\phi - \frac{1}{2}\log(r)$ is a decreasing function of ϕ for $0 < \phi < \pi$, the major contribution of this arc is bounded above by the value of the integrand at the end point where $\rho = n^{\delta}$. A deformation of the path of integration near -1 is immaterial so the estimate of the remainder of the path is again bounded above by the value of the integrand at $\rho = n^{\delta}$. (Therefore in the next section we will focus on the asymptotics on the line segment in the first quadrant only.)

If we set $z = re^{i\theta}$ where $r^2 = 1 + 2^{\frac{2}{3}} l_n^{-\frac{4}{3}} n^{2\delta} - 2^{\frac{1}{3}} l_n^{-\frac{2}{3}} n^{\delta}$ and $\theta_0 \le \theta \le \pi - \theta_0$ where θ_0 is the angle that the ray from the origin to the tip of the line segment makes with the real axis, $\cos(\theta_0) = \frac{2 - 2^{\frac{1}{3}} l_n^{-\frac{2}{3}} n^{\delta}}{2r}$. The modulus of this integrand in (A.5) is of order

$$r^{c-\frac{3}{2}} e^{k(\theta)} \quad \text{where} \quad k(\theta) = -\frac{1}{4} r^2\, \cos(2\theta) + r\, \cos(\theta) - \frac{1}{2}\log(r) + 2^{-\frac{1}{3}} l_n^{-\frac{4}{3}}\, r\, t \cos(\theta)$$

$k^{'}$ has roots $\pi,\ 0$ and θ_2 where $\cos(\theta_2) = \frac{1 + 2^{-\frac{1}{3}} l_n^{-\frac{4}{3}} t}{r}$, for large $l_n,\ 0 \le \theta_2 \le \theta_0$. k increases from 0 to θ_2 and decreases from θ_2 to π. Therefore the maximum of $k(\theta)$ on the arc of circle is at θ_0. We can also use this as an upper bound of the contribution of the contour around -1. Thus the contribution of the arc of circle and the line segment on the second quadrant is of order of the modulus of the integrand evaluated at the tip of the line segment in the first quadrant.We can therefore focus our attention in the next section on just the line segment in the first quadrant. The parametrization (A.8) shows that ρ will go to infinity with n, so we want to estimate the contribution not only of the line segment but of the whole ray $0 \le \rho \le \infty$. The error that we make by taking the ray is $|\int_{n^{\delta}}^{\infty} z^{c-\frac{2}{3}} e^{-l_n^2 f_1(z) + 2^{-\frac{1}{3}} l_n^{\frac{2}{3}} tz} dz|$.

[9] See [18, Section 8.75] for an illustration of this path.

On the ray, $-l_n^2 f_1(z) = \frac{3}{4}l_n^2 - \frac{\rho^3}{3} + O(\rho^4) l_n^{-\frac{2}{3}}$, thus
$\int_{n^\delta}^\infty z^{c-\frac{2}{3}} e^{-l_n^2 f_1(z) + 2^{-\frac{1}{3}} l_n^{\frac{2}{3}} tz} dz = O(1) 2^{\frac{1}{3}} l_n^{-\frac{2}{3}} e^{\frac{3}{4}l_n^2 + 2^{-\frac{1}{3}} l_n^{\frac{2}{3}} t} \int_{n^\delta}^\infty e^{-\frac{\rho^3}{3} + \rho t e^{\frac{2\pi i}{3}}} d\rho$.
We therefore need to give an estimate an estimate of the integral on the right of this last equality and an estimate of the integrand when $\rho = n^\delta$ in (A.5).

Integrand when $\rho = n^\delta$

The order of the integrand for large n is $O(1) 2^{\frac{1}{3}} l_n^{-\frac{2}{3}} e^{\frac{3}{4}l_n^2 + 2^{-\frac{1}{3}} l_n^{\frac{2}{3}} t} e^{-\frac{\rho^3}{3} + \rho t e^{\frac{2\pi i}{3}}}$.
We need to give an estimate of the factor $u := e^{-\frac{\rho^3}{3} + \rho t e^{\frac{2\pi i}{3}}}$ as a function of t.
If t is positive, it is of order $e^{-\frac{n^{3\delta}}{3}} \cdot e^{-\frac{tn^\delta}{2}}$ or of order $e^{-\frac{n^{3\delta}}{3}} \cdot e^{-t}$.
If t is negative, we have $|u| = e^{-\frac{\rho^3}{3} + \frac{\rho y}{2}}$ where $y = -t$. For trace class convergence we need an estimate that will decay exponentially for large t. The expansion of the exponent in $|u|$ around his critical point $\rho_0 = -\sqrt{\frac{y}{2}}$ is $-\frac{2}{3}(\frac{y}{2})^{\frac{3}{2}} + \sqrt{\frac{y}{2}}(\rho + \sqrt{\frac{y}{2}})^2 - \frac{1}{3}(\rho + \sqrt{\frac{y}{2}})^3$.
Thus for $\rho = n^\delta$, u is of order $e^{-\frac{(n^\delta + \sqrt{\frac{y}{2}})^3}{3}} \cdot e^{-\frac{2}{3}(\frac{y}{2})^{\frac{3}{2}}}$ or of order $e^{-\frac{n^{3\delta}}{3}} \cdot e^{-y}$ since y is bounded and positive. So in either case the contribution is of order, $e^{-\frac{n^{3\delta}}{3}} \cdot e^{-|t|}$.

Tail integration

We show that the contribution of $\int_{n^\delta}^\infty e^{-\frac{\rho^3}{3} - \frac{\rho t}{2}} d\rho$ is of the same order.

If $t \geq 0$, $\int_{n^\delta}^\infty e^{-\frac{\rho^3}{3} - \frac{\rho t}{2}} d\rho \leq \int_{n^\delta}^\infty e^{-\frac{\rho^3}{3} - \frac{t}{2}} d\rho = e^{-\frac{t}{2}} \int_{n^\delta}^\infty e^{-\frac{\rho^3}{3}} d\rho \leq e^{-\frac{t}{2}} \int_{n^\delta}^\infty e^{-\frac{\rho}{3}} d\rho$
so it is of order $e^{-\frac{n^\delta}{3}} \cdot e^{-\frac{t}{2}}$
If t is negative, a similar change of variable and expansion of the integrand leads to
$\int_{n^\delta}^\infty e^{-\frac{\rho^3}{3} - \frac{\rho t}{2}} d\rho \quad = \int_{n^\delta}^\infty e^{-\frac{2}{3}(\frac{y}{2})^{\frac{3}{2}} + \sqrt{\frac{y}{2}}(\rho + \sqrt{\frac{y}{2}})^2 - \frac{1}{3}(\rho + \sqrt{\frac{y}{2}})^3} d\rho$
$e^{-\frac{2}{3}(\frac{y}{2})^{\frac{3}{2}}} \int_{n^\delta}^\infty e^{-\frac{1}{3}(\rho + \sqrt{\frac{y}{2}})^3 [1 - 3(\rho + \sqrt{\frac{y}{2}})^{-1} \sqrt{\frac{y}{2}}]} d\rho \quad \sim e^{-\frac{2}{3}(\frac{y}{2})^{\frac{3}{2}}} \int_{n^\delta}^\infty e^{-\frac{1}{3}(\rho + \sqrt{\frac{y}{2}})^3} d\rho$
This is of order, $e^{-\frac{2}{3}(\frac{y}{2})^{\frac{3}{2}}} \int_{n^\delta}^\infty e^{-\frac{1}{3}(\rho + \sqrt{\frac{y}{2}})} d\rho$ or of order $e^{-\frac{2}{3}(\frac{y}{2})^{\frac{3}{2}}} \cdot e^{-\frac{1}{3}(n^\delta + \sqrt{\frac{y}{2}})}$
So the contribution of this integral is of order $e^{-\frac{n^\delta}{3}} \cdot e^{-|t|}$

We conclude this subsection by recording that the error that we make by neglecting the remainder of the contour and considering the integral from zero to infinity instead of zero to n^δ is at most of order

$$2^{\frac{1}{3}} l_n^{-\frac{2}{3}} e^{\frac{3}{4}l_n^2 + 2^{-\frac{1}{3}} l_n^{\frac{2}{3}} t} e^{-\frac{n^\delta}{3}} \cdot e^{-|t|} \tag{A.9}$$

A.1.2 On the ray $z = 1 + 2^{\frac{1}{3}} l_n^{-\frac{2}{3}} \rho e^{\frac{2\pi i}{3}}$, $\quad \rho \in [0, \infty)$

The Taylor expansion of f_1 at $z_0 = 1$ is
$f_1(z) = -\frac{3}{4} + \frac{\rho^3}{3} l_n^{-2} - \frac{1}{2} \sum_{k=4}^\infty c_{1,k} (\rho l_n^{-\frac{2}{3}})^k$ with $c_{1,k} = (-1)^k \frac{2^{\frac{k}{3}} e^{\frac{2\pi k i}{3}}}{k}$, and

$2^{-\frac{1}{3}}l_n^{\frac{2}{3}}tz = 2^{-\frac{1}{3}}l_n^{\frac{2}{3}}t + \rho t e^{\frac{2\pi i}{2}}$. Taking in account the error estimate from (A.9), the substitution of these quantities in G give,

$$G = 2^{\frac{1}{3}}l_n^{-\frac{2}{3}}e^{\frac{2\pi i}{3}}e^{\frac{3}{4}l_n^2+2^{-\frac{1}{3}}l_n^{\frac{2}{3}}t}\cdot\left\{\int_0^\infty e^{-\frac{\rho^3}{3}+\rho t e^{\frac{2\pi i}{3}}}g_n(\rho)\,d\rho \quad + O(e^{-\frac{n^\delta}{3}})\cdot e^{-|t|}\right\} \quad \text{with}$$

$$g_n(\rho) = \left(1 - \frac{4\alpha^2-1}{8l_n^2(1+2^{\frac{1}{3}}l_n^{-\frac{2}{3}}\rho e^{\frac{2\pi i}{3}})(1+2^{-\frac{1}{3}}l_n^{-\frac{4}{3}}t)}\right)(1+2^{\frac{1}{3}}l_n^{-\frac{2}{3}}\rho e^{\frac{2\pi i}{3}})^{c-\frac{3}{2}}e^{\frac{1}{2}\sum_{k=4}^\infty c_{1,k}(\rho l_n^{-\frac{2}{3}})^k l_n^2} \tag{A.10}$$

A.2 Steepest descent for H

The analysis for H differs from that of G in the location of the saddle point, and the orientation of the contour. A similar analysis shows that the saddle point is now at $z_0 = -1$, the final contour of integration is the same but oriented in the opposite direction. It leaves z_0 at angle $\pi/3$. The error estimate on the arc of circle and on the tail of the corresponding ray is the same. The new parametrization on the ray is $z = -1 + 2^{\frac{1}{3}}l_n^{-\frac{2}{3}}e^{\frac{\pi i}{3}}\rho = -(1 + 2^{\frac{1}{3}}l_n^{-\frac{2}{3}}e^{-\frac{2\pi i}{3}}\rho), \quad 0 \le \rho \le \infty$
The Taylor expansion of f_2 at $z = -1$ is

$$f_2(z) = -\frac{3}{4} + \frac{\pi}{2}i + l_n^{-2}\frac{\rho^3}{3} - \frac{1}{2}\sum_{k\ge 4} c_{2,k}\rho^k l_n^{-\frac{2k}{3}}, \quad \text{with} \quad c_{2,k} = (-1)^k\frac{2^{\frac{k}{3}}e^{-\frac{2\pi k i}{3}}}{k}$$

This leads to

$$H = -2^{\frac{1}{3}}l_n^{-\frac{2}{3}}e^{-\frac{2\pi i}{3}}e^{\frac{3}{4}l_n^2+2^{-\frac{1}{3}}l_n^{\frac{2}{3}}t}\cdot\left\{\int_0^\infty e^{-\frac{\rho^3}{3}+\rho t e^{-\frac{2\pi i}{3}}}h_n(\rho)\,d\rho \quad + O(e^{-\frac{n^\delta}{3}})\cdot e^{-|t|}\right\} \quad \text{with}$$

$$h_n(\rho) = \left(1 - \frac{4\alpha^2-1}{8l_n^2(1+2^{\frac{1}{3}}l_n^{-\frac{2}{3}}\rho e^{-\frac{2\pi i}{3}})(1+2^{-\frac{1}{3}}l_n^{-\frac{4}{3}}t)}\right)(1+2^{\frac{1}{3}}l_n^{-\frac{2}{3}}\rho e^{-\frac{2\pi i}{3}})^{c-\frac{3}{2}}e^{\frac{1}{2}\sum_{k=4}^\infty c_{2,k}(\rho l_n^{-\frac{2}{3}})^k l_n^2} \tag{A.11}$$

A.3 Asymptotics for K

The asymptotics of the integral factor in **K** depends on the leading term in the expansion of either **G** or **H**, depending on which one is larger as shown in (A.7). But from the previous analysis, the leading term of both **G** and **H** are of the same order. Thus **K** is also of order of a nonzero linear combination of the leading terms in **G** and in **H** times $O(l_n^{-4})$. In our case[10] we take the linear combination to be $\mathcal{R}e\frac{1}{2\pi i}(\mathbf{G}+\mathbf{H})$.

[10] The choice of this representation of the error is for trace class convergence of the final result.

A.4 Conclusion

Note that from (A.10) and (A.11) we see that except for the O-term, $H = -\overline{G}$. The change of variable $\rho \mapsto \rho e^{-\frac{2\pi i}{3}}$ transform $g_n(\rho)$ into a real function $g_1(\rho)$ and

$$\mathbf{G} = 2^{\frac{1}{3}} l_n^{-\frac{2}{3}} e^{\frac{3}{4}l_n^2 + 2^{-\frac{1}{3}} l_n^{\frac{2}{3}} t} \cdot \left(\int_0^{\infty e^{\frac{2\pi i}{3}}} e^{(-\frac{\rho^3}{3} + \rho t)} g_1(\rho) d\rho + O(e^{-\frac{n^\delta}{3}}) e^{-|t|} \right), \quad \text{note also that}$$

$$2\mathcal{R}e \frac{1}{2\pi i}(G + H) = 2 \cdot 2^{\frac{1}{3}} l_n^{-\frac{2}{3}} e^{\frac{3}{4}l_n^2 + 2^{-\frac{1}{3}} l_n^{\frac{2}{3}} t}.$$

$$\left\{ \frac{1}{2\pi i} [\int_{\infty e^{-\frac{2\pi i}{3}}}^0 e^{(-\frac{\rho^3}{3} + \rho t)} g_1(\rho) d\rho + \int_0^{\infty e^{\frac{2\pi i}{3}}} e^{(-\frac{\rho^3}{3} + \rho t)} g_1(\rho) d\rho] + O(e^{-\frac{n^\delta}{3}}) e^{-|t|} \right\} \tag{A.12}$$

At this point we can give an estimate for $\mathbf{K}$ based on this last formula since the leading term of g_1 is 1.

$$\mathbf{K} = O(l_n^4) \cdot 2^{\frac{1}{3}} l_n^{-\frac{2}{3}} e^{\frac{3}{4}l_n^2 + 2^{-\frac{1}{3}} l_n^{\frac{2}{3}} t} \cdot \frac{1}{2\pi i} \left(\int_{\infty e^{-\frac{2\pi i}{3}}}^0 e^{(-\frac{\rho^3}{3} + \rho t)} d\rho + \int_0^{\infty e^{\frac{2\pi i}{3}}} e^{(-\frac{\rho^3}{3} + \rho t)} d\rho \right)$$

$$= \quad O(l_n^{-4}) \cdot 2^{\frac{1}{3}} l_n^{-\frac{2}{3}} e^{\frac{3}{4}l_n^2 + 2^{-\frac{1}{3}} l_n^{\frac{2}{3}} t} \cdot \mathrm{Ai}(t) \tag{A.13}$$

where $\quad \mathrm{Ai}(t) = \frac{1}{2\pi i} \left(\int_{\infty e^{-\frac{2\pi i}{3}}}^0 e^{(-\frac{\rho^3}{3} + \rho t)} d\rho + \int_0^{\infty e^{\frac{2\pi i}{3}}} e^{(-\frac{\rho^3}{3} + \rho t)} d\rho \right)$ is the Airy function (A.14)

To simplify notation, we will combine the two paths of integration in $\mathbf{G}$ and call the new path σ.

With this notation, (A.4) gives

$$e^{-\frac{\xi^2}{2}} L_n^\alpha(\xi^2) = (-1)^n 2^{2n} \Gamma(n + \alpha + 1) 2^\alpha (\xi)^{-\alpha} e^{-\frac{\xi^2}{2}} l_n^{-2n-\alpha} (2\pi \xi l_n)^{-\frac{1}{2}} 2 \cdot 2^{\frac{1}{3}} l_n^{-\frac{2}{3}} e^{\frac{3}{4}l_n^2 + 2^{-\frac{1}{3}} l_n^{\frac{2}{3}} t}.$$

$$\left[\frac{1}{2\pi i} \int_\sigma e^{(-\frac{\rho^3}{3} + \rho t)} g_1(\rho) d\rho \quad + \quad O(l_n^{-4}) Ai(t) \quad + \quad O(e^{-\frac{n^\delta}{3}}) e^{-|t|} \right] \tag{A.15}$$

Using (A.10) and the help of *Mathematica*, we have the following expansion[11] of g_1 in powers of $l_n^{-\frac{2}{3}}$.

$$1 \; + \; 2^{\frac{1}{3}} \left(\frac{\rho^4}{4} + (c - \frac{3}{2})\rho \right) l_n^{-\frac{2}{3}} \; + 2^{\frac{2}{3}} \left(-\frac{\rho^5}{5} + \frac{\rho^8}{32} + \frac{(c - \frac{3}{2})\rho^5}{4} + \frac{(c - \frac{3}{2})\rho^2 (c - \frac{5}{2})}{2} \right) l_n^{-\frac{4}{3}}$$

[11] The expansion is valid for this derivation up to $l_n^{-6\frac{2}{3}}$ as the error estimate for K indicates. We choose to stop here at $l_n^{-4\frac{2}{3}}$ note also that the last term will served to estimate the error.

$$+\left(\frac{(c-\frac{3}{2})\rho^6(c-\frac{5}{2})}{4}+\frac{\rho^6}{3}-\frac{\rho^9}{10}+\frac{\rho^{12}}{192}+\frac{1}{3}(c-\frac{3}{2})\rho^3(c-\frac{5}{2})(c-\frac{7}{2}) \quad - \right.$$
$$\left.\frac{a^2}{2}+\frac{1}{8}+(c-\frac{3}{2})2\rho(-\frac{\rho^5}{5}+\frac{\rho^8}{32})\right) l_n^{-2} \quad +$$
$$\left(2^{\frac{1}{3}}(-\frac{2\rho^7}{7}+\frac{\rho^{10}}{12}+\frac{\rho^{10}}{25}-\frac{\rho^{13}}{80}+\frac{\rho^{16}}{3072})+\frac{1}{4}\left(\frac{(c-\frac{3}{2})\rho^3(c-\frac{5}{2})(c-\frac{7}{2})}{3}-\frac{a^2}{2}+\frac{1}{8}\right)\sqrt[3]{2}\rho^4\right.$$
$$+(c-\frac{3}{2})2^{\frac{1}{3}}\rho^2(c-\frac{5}{2})(-\frac{\rho^5}{5}+\frac{\rho^8}{32})+(c-\frac{3}{2})2^{\frac{1}{3}}\rho(\frac{\rho^6}{3}-\frac{\rho^9}{10}+\frac{\rho^{12}}{192})$$
$$\left.+\frac{2^{\frac{1}{3}}}{12}(c-\frac{3}{2})\rho^4(c-\frac{5}{2})(c-\frac{7}{2})(c-\frac{9}{2})+(-\frac{a^2}{2}+\frac{1}{8})(c-\frac{3}{2})2^{\frac{1}{3}}\rho-\frac{1}{8}(-4\,a^2+1)2^{\frac{1}{3}}\rho\right) l_n^{-8/3}$$
$$+q(\rho)O(l_n^{-\frac{10}{3}}) \quad \text{for some polynomial } q$$

The integral in (A.15) can be expresses as a linear combination of the Airy function and its derivative using $\mathrm{Ai}^{(k)}(t)=\frac{1}{2\pi i}\int_\sigma e^{-\frac{\rho^3}{3}+\rho t}\rho^k d\rho$ and this expansion of g_1. Using the Airy differential equation $\mathrm{Ai}''(t)=t\,\mathrm{Ai}(t)$, it reduces to an expression involving only the independent variable t, Ai and Ai$'$.

The contribution of the last term of the expansion of g_1 is a finite combination of the form $\sum_{m=0}^k (p_m(t)\ \mathrm{Ai}(t)+q_m(t)\ \mathrm{Ai}'(t))$ for some polynomials p and q. If t is bounded away from minus infinity, this is of order $\mathrm{Ai}(t)$. In this paper we assume therefore that this is the case for t.

If $\xi=l_n+(2l_n)^{-\frac{1}{3}}t$ we have

$e^{-\frac{\xi^2}{2}}\xi^{-(\alpha+\frac{1}{2})} = l_n^{-(\alpha+\frac{1}{2})}e^{-\frac{l_n^2}{2}-2^{-\frac{1}{3}}l_n^{\frac{2}{3}}t}[1-\frac{t^2}{2^2}2^{\frac{1}{3}}l_n^{-\frac{2}{3}}+(\frac{t^4}{2^5}-(2\alpha+1)\frac{t}{2^2})2^{\frac{2}{3}}l_n^{-\frac{4}{3}}+(-\frac{t^6}{3\cdot 2^6}+(2\alpha+1)\frac{t^3}{2^3})l_n^{-2}+O(l_n^{-\frac{8}{3}}t^2)]$.

Stirling formula gives $\Gamma(n+\alpha+1)=2^{\frac{1}{2}}\pi^{\frac{1}{2}}n^{n+\alpha+\frac{1}{2}}e^{-n}[1+\frac{6\alpha(\alpha+1)+1}{12n}+O(n^{-2})]$ and we have $l_n^{-2n-2\alpha-\frac{5}{3}}=2^{-2n-2\alpha-\frac{5}{3}}n^{-n-\alpha-\frac{5}{6}}e^{-\frac{\alpha+c}{2}}[1+\frac{(\alpha+c)(-9\alpha+3c-10)}{24n}+O(n^{-2})]$.

Thus

$(-1)^n2^{2n}\Gamma(n+\alpha+1)2^\alpha(\xi)^{-\alpha}l_n^{-2n-\alpha}(2\pi\xi l_n)^{-\frac{1}{2}}2\cdot 2^{\frac{1}{3}}l_n^{-\frac{2}{3}}e^{\frac{3}{4}l_n^2+2^{-\frac{1}{3}}l_n^{\frac{2}{3}}t}e^{-\frac{\xi^2}{2}} =$

$(-1)^n2^{-\alpha-\frac{1}{3}}n^{-\frac{1}{3}}(1+\frac{3\alpha^2+2\alpha-6\alpha c+3c^2-10c+2}{24n}+O(n^{-2}))[1-\frac{t^2}{2^2}2^{\frac{1}{3}}l_n^{-\frac{2}{3}}+(\frac{t^4}{2^5}-(2\alpha+1)\frac{t}{2^2})2^{\frac{2}{3}}l_n^{-\frac{4}{3}}+(-\frac{t^6}{3\cdot 2^6}+(2\alpha+1)\frac{t^3}{2^3})l_n^{-2}+O(l_n^{-\frac{8}{3}}t^2)]$.

This in (A.15) together with the expansion of g_1 give the desired result.[12]

[12]Actually

$$\xi=(4n+2\alpha+2c)^{\frac{1}{2}}+\frac{t}{2^{\frac{2}{3}}n^{\frac{1}{6}}}+O(n^{-\frac{7}{6}}),$$

but due to the smoothness of $e^{-\frac{\xi^2}{2}}L_n^\alpha(\xi^2)$ the error that we make by removing the O-term is negligible if we are aiming for an accuracy of order n^{-1}.

For $\xi = (4n+2\alpha+2c)^{\frac{1}{2}} + \frac{t}{2^{\frac{2}{3}} n^{\frac{1}{6}}}$ and t bounded,

$$e^{-\frac{\xi^2}{2}} L_n^\alpha(\xi^2) = (-1)^n 2^{-\alpha-\frac{1}{3}} n^{-\frac{1}{3}} \{\mathrm{Ai}(t) + \frac{(c-1)}{2^{\frac{1}{3}}} \mathrm{Ai}'(t) n^{-\frac{1}{3}} +$$

$$[\frac{2-10c+5c^2-5\alpha}{10\cdot 2^{\frac{2}{3}}} t\,\mathrm{Ai}(t) + \frac{t^2}{20\cdot 2^{\frac{2}{3}}} \mathrm{Ai}'(t)] n^{-\frac{2}{3}} +$$

$$[(\frac{5\alpha - 15c\alpha + 2c^3 - 15c^2 - 56c - 6}{60} + \frac{c-1}{40} t^3)\,\mathrm{Ai}(t)$$

$$+\frac{(c-1)(5(c-2)c-3(2+5\alpha))}{60} t\,\mathrm{Ai}'(t)] n^{-1} + O(n^{-\frac{4}{3}})\,\mathrm{Ai}(t) \quad \} \qquad \text{(A.16)}$$

This is the desired formula for this section. Figure A.4 gives an illustration of our asymptotics.

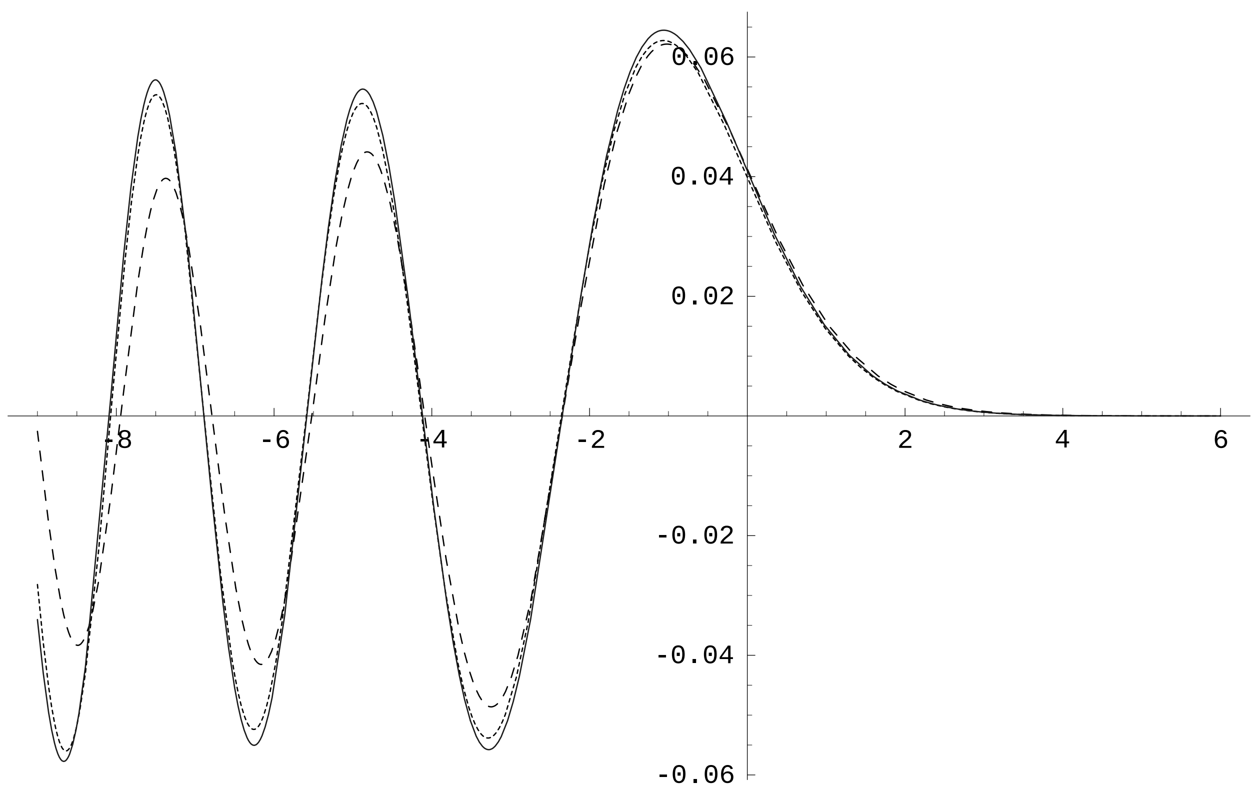

Figure A.4.1: For $\alpha = -c = 1$ and $n = 40$, the solid curve represents $e^{-\frac{\xi^2}{2}} L_n^\alpha(\xi^2)$, the dashed curve is the usual first order approximation of $e^{-\frac{\xi^2}{2}} L_n^\alpha(\xi^2)$ in term of the Airy function (Which is the first term approximation in (A.16)), the doted curve represents our approximation. These are functions of t where $\xi = (4n+2\alpha+2c)^{\frac{1}{2}} + \frac{t}{2^{\frac{2}{3}} n^{\frac{1}{6}}}$

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