CAL III: DOUBLE INTEGRALS & TRIPLE INTEGRALS

Double Integrals:

Traditionally, volume can be calculated by adding slices of area: $V = \int_{x_0}^{x_f} A(x)dx$

- However, $A(x)$ can be written as $A(x) = \int_{y_0}^{y_f} z(x, y)dy$, where z is a surface as a function of x and y.
- The volume can be written as a double integral

$$V = \int_{x_0}^{x_f}\int_{y_0}^{y_f} z(x, y)dydx$$

- Note that multiple integrals have pairs of bounds and differentials, which are evaluated from the innermost to the outermost pair. However, the order of these pairs may be rearranged, just as with partial differentiation:

$$\int_c^d\int_a^b f(x, y)dxdy = \int_a^b\int_c^d f(x, y)dydx \quad for \quad a\leq x\leq b \text{ and } c\leq y\leq d \quad (Fubini's\ Theorem)$$

- The bounds of integration are not confined to constants; they will often be functions of x and y.

- Finding limits of integration can be done using cross-sections: sketch an x-y plane with the shadow of the volume at hand. For y limits, draw a line L parallel to the y-axis and determine the functions at which it enters the region and which it exits. Likewise for x limits, use a line L parallel to x-axis.

- Note that the outermost/last variable of integration should be a constant so that the final answer will be a constant. In the case when L determines the y limits of integration, the x limits should be two constants that bound all the parallel lines to L that enter and exit the region.

- Likewise, area may be found with a double integral. In this case, there is not a third dimension to add height to the object: $z(x, y) = 1$ such that the double integral becomes:

$$A = \int_{x_0}^{x_f}\int_{y_0}^{y_f} 1\cdot dtdx$$

Triple Integrals:

- Volume can be found using the shadow of a four-dimensional region when the function $F(x, y, z) = 1$ over the region D:

$$V = \iiint_D 1 \cdot dV = \iiint_D dz\, dy\, dx$$

- Determining the limits of integration follow the same rules as for double integrals. Again, the final variable (in the example above, x) to be integrated should have constant limits. The inner variable may be functions of the other variables, as determined by sketching a x-y plane and tracing a line L parallel to the axis of the variable being evaluated.

Partial Derivatives:

- Partial Derivatives hold all but one variable constant and differentiate that variable. Each partial tells us the rate of change *with respect to the associated variable* $\frac{\partial f}{\partial x} = f_x$ and $\frac{\partial f}{\partial y} = f_y$ for $f = f(x, y)$. The normal $\frac{d}{d()}$ is replaced with the curved/cursive version (not associated with any particular language) to signify partial, not total: $\frac{\partial}{\partial()}$

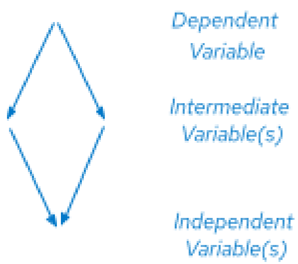
- Second order differentials can be found analogously to one-dimensional second order derivatives, but care must be taken as to which variable is evaluated

Note that order of operations does not affect end result

$$f_x = \frac{\partial f}{\partial x} \quad , \quad f_y = \frac{\partial f}{\partial y} \quad , \quad f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \quad , \quad f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \quad , \quad f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

- The Chain Rule is useful and used in a similar fashion to single variable functions:

- If w is a function of x and y, which are a function of t, then $\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dy}$



This tree diagram is scalable for any number of intermediate and independent variables. Add across branches and multiply down the paths. Derivatives connect each layer.