Cal III: Line Integrals & Vector Fields

Line Integrals and Vector Fields:

- Line integrals are used as a general form of integration over a curve C rather than an interval. The curve needs to be parameterized using a ray that traces location as a function of t:

$$r(t) = g(t)i + h(t)j + k(t)k \quad for \ \ a \leq t \leq b \quad such\ that \ \ f(x, y, z) = f(g(t), h(t), k(t))$$

- Knowing that $\left|\frac{ds}{dt}\right| = |v(t)|$, a line integral is written:

$$\int_C f(x, y, z)ds = \int_a^b f(g(t), h(t), k(t)) \cdot |v(t)|dt$$

- Line integrals can be used in vector fields to find work, flux, and more. A vector field is defined:

$$F(x, y, z) = M(x, y, z)i + N(x, y, z)j + P(x, y, z)k$$

- One notable vector field is the gradient field, defined by the gradient vector F of a scalar function f**:**

$$\nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$$

- Knowing that the tangent vector is defined as $T = \frac{dr}{ds}$, which defines the forward motion of the path, the line integral of a vector field F over path $r(t)$ can be written:

$$\int_C F \cdot T\, ds = \int_C \left(F \cdot \frac{dr}{ds}\right)ds = \int_C F \cdot dr = \int_a^b F(r(t)) \cdot \frac{dr}{dt}dt$$

- It is known that this integral is equal to the work done by a force F over a curve C from a to b as well as the flow of a fluid along the curve C.

- The flux of F across the curve C is defined by the scalar component of the fluid's velocity in the direction of the curve's *outward facing normal vector* (while the tangential vector leads to flow **along** the curve, flux is concerned with flow **across** the curve).

$$Flux = \int_C F \cdot n\, ds = \int_C F \cdot (T \times k)\, ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds = \oint_C M\, dy - N\, dx$$