CAL III: POLAR, CYLINDRICAL, & SPHERICAL COORDINATES

Polar and Cylindrical Coordinates:

Jacobian of the transformation

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\cdot r\, dr\, d\theta$$

- For limits of integration for a polar double integral, draw a ray extending radially outward from the origin and determine the functions at which it enters and at which it exits. For the θ limits, determine the initial and final angle of this ray. *Note: θ will usually be the last variable of integration.*

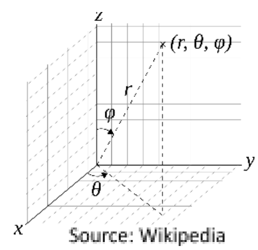
- The Cylindrical coordinate system is built on the polar coordinate system with the addition of a variable to describe the distance from the r-θ plane (thought of as height"), notated as z. Note that this z is equivalent to that of the rectangular coordinate z.

Jacobian of the transformation

$$\iiint_D f\, dV = \iiint_D f\, dz\, r\, dr\, d\theta$$

- The limits of integration are similar to polar for r and θ and to rectangular for z. *Note: θ will usually be the last variable of integration.*

Spherical Coordinates:



Source: Wikipedia

- Spherical coordinates are defined by three parameters:
  1) ρ, the radial distance from a point to the origin.
  2) ϕ, the polar angle between a point and the positive z-axis.
  3) θ, the azimuth angle between the shadow of ρ on the x-y plane and positive x-axis.

Jacobian of the transformation

$$\iiint_D f(x, y, z)dV = \iiint_D f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\theta)dV = \iiint_D f(\rho,\ \phi,\ \theta)\ \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

- The limits of integration are found in similar fashion as cylindrical. Typically, first find ρ limits by drawing a ray from the origin through the region at angle ϕ and observe the functions which it enters and exits. Next, determine ϕ limits by observing the minimum (≤0°) and maximum (≥180°) angle made with the positive z-axis. Finally, observe the "shadow" of ray ρ on the x-y plane and determine the minimum and maximum angles it makes with the positive x-axis as it sweeps through the entire region.

Cylindrical to Rectangular	**Spherical to Rectangular**	**Spherical to Cylindrical**
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$
$dV =$	$dx\,dy\,dz$	Rectangular
	$dz\,r\,dr\,d\theta$	Cylindrical
	$\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$	Spherical

Change of Variables:

- A change of variables or coordinate systems is often useful in solving complex integrals. When doing so, a Jacobian is a necessary accompaniment:

$$J(u,\ v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \cdot \frac{\partial x}{\partial v} = \frac{\partial(x,y)}{\partial(u,v)}$$

- For example, the following Jacobian is required to transform cartesian to polar coordinates:

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial(r\cos\theta)}{\partial r} & \frac{\partial(r\cos\theta)}{\partial\theta} \\ \frac{\partial(r\sin\theta)}{\partial r} & \frac{\partial(r\sin\theta)}{\partial\theta} \end{vmatrix} = |\cos\theta\, r\cos\theta\ +\ r\sin\theta\, \sin\theta\,| = r(\theta\ +\ \theta\,)\ =\ r$$

- Likewise, it can be shown that $J(\rho, \phi,\ \theta) = \rho^2 \sin\phi$ by adding a third row $\left(\frac{\partial z}{\partial()}\right)$ and third column $\left(\frac{\partial()}{\partial\phi}\right)$.